Environmental policy, education and growth: A reappraisal when lifetime is finite

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Abstract

This article demonstrates that when finite lifetime is introduced in a Lucas (1988) growth model where the source of pollution is physical capital, the environmental policy may enhance the growth rate of a market economy, while pollution does not influence educational activities, labor supply is not elastic and human capital does not enter the utility function. The result arises from the "*generational turnover effect*" due to finite lifetime and it remains valid under conditions when the education sector uses final output besides time to accumulate human capital.

This article also demonstrates that ageing reduces the positive influence of the environmental policy when growth is driven by human capital accumulation à la Lucas (1988) in the Yaari (1965)-Blanchard (1985) overlapping generations model

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1 Introduction

While the link between environment and growth has already been extendedly investigated (see Xepapadeas et al., 2005; Brock and Taylor, 2005), conclusions about the influence of the environmental policy on economic growth remain open. The purpose of this article is to contribute to the debate, by re-examining the influence of the environmental policy on human capital based-growth when finite lifetime is taken into account. It demonstrates that finite lifetime introduces a new channel of transmission between the environment and economic performances based on the turnover of generations.

Most of the industrialized countries are now becoming knowledge- and education-based economies using more and more human capital instead of physical capital to produce. And education played a major role in the industrialization of the South-East Asian countries during 7os and 80s decades.[1] Nevertheless, few theoretical works investigate environmental issues in a framework where human capital is the engine of growth and economic prosperities.[2] A noteworthy exception is the seminal article by Gradus and Smulders (1993). In a model à la Lucas (1988) where pollution originates from physical capital, they demonstrate that the environment never influences the steady-state growth rate except when pollution affects education activities.[3] This result comes from the fact that the growth rate of consumption relies on the after-tax interest rate and the rate of time preferences and that the tax-rate is invariant with pollution tax in the steady-state when labor supply is inelastic. When the environmental taxation increases,

[1]See World Bank (1993) for empirical evidence of this role. Grimaud and Tournemaine (2007) point out this role to justify the need to investigate the link between environment and growth through the channel of education.

[2]Here, we do not deal with the major question of climate change and we do not integrate non-renewable resources in the analysis. See Schou (2000, 2002) and Grimaud and Rouge (2005, 2008) for authors who investigate environmental policy and growth in the presence of non-renewable resources and endogenous growth. Note also that there exists an important literature on the impact of environmental policy on growth, even if the contributions do not deal with human capital accumulation: on the double dividend, see Bovenberg and Smulders (1995, 1996), Bovenberg and de Mooij (1997); for contributions using OLG model, see Ono (2002, 2003)(discrete time model) or Bovenberg and Heijdra (1998, 2002) (continuous time Yaari-Blanchard model) amongst others.

[3]More precisely, they assume that pollution depreciates the stock of human capital. van Ewijk and van Wijnbergen (1995) consider that pollution reduces the ability to train .

the after-tax interest rate reduces and becomes lower than the returns to human capital. Consequently, the investment in physical capital drops in favor to human capital accumulation. Final production becomes more intensive in human capital and the allocation of human capital in production diminishes. This mechanism perpetuates until the after-tax interest rate backs to its initial value equal to the rate of returns in human capital accumulation. Because the aggregate consumption growth in the steady-state relies only on the after-tax interest rate, it is not modified by the higher pollution tax.

Assuming that labor supply is elastic and pollution originates from the stock of physical capital, Hettich (1998) finds a positive influence of the environmental policy on human capital accumulation, in a Lucas' setting. The increase in the environmental tax compels firms to increase their abatement activities at the expense of the household's consumption. To counteract this negative effect, households substitute leisure to education and the growth rate rises. Nevertheless, the author demonstrates that his result is very sensitive to the assumption about the source of pollution. When pollution originates final output rather than physical capital, the link between the environment and growth does not longer exist. By taxing output, a tighter environmental policy reduces both the returns to physical capital and the wage rate which contributes to the returns to education. The incentives of agents to invest more in education vanish.

More recently, Grimaud and Tournemaine (2007) demonstrate that a tighter environmental policy promotes growth, in a model combining R&D and human capital accumulation, where education directly enters the utility function as a consumption good and knowledge from R&D reduces the flow of pollution emissions. By increasing the price of the good whose production pollutes the higher tax rate reduces the relative cost of education and therefore incites agents to invest in human capital accumulation. Because education is the engine of growth, the growth rises in the steady-state. As highlighted by the authors, the key assumption is the introduction of the education as a consumption good in utility which lets the returns to education dependent from the environmental policy. When education does not influence utility, the returns to education is exogenous and therefore is not affected by the policy.

In the present article, we re-examine the link between the environmental policy and growth in a Lucas' setting, assuming just that lifetime is finite. We use a Yaari (1965)-Blanchard (1985) overlapping generations model where growth is driven by human capital accumulation à la Lucas (1988) and pollution arises from physical capital.[4] We study both the long-run balanced growth path and the transition.

Our results are as follows. First, we demonstrate that when lifetime is finite and physical capital is the source of pollution, a tightening environmental policy enhances growth even if pollution does not affect educational activities, labor supply is inelastic and human capital does not enter the utility function. Indeed, besides the crowding-out effect and the substitution effect of a tighter environmental tax on growth (see Gradus and Smulders, 1993), when agents have finite lifetime there exists a third impact arising from the turnover of generations. Because at each date a new generation is born without financial assests and a cross-section of the existing population dies, the aggregate consumption rate of growth is enhanced by a higher environmental tax and agents invest more in human capital (the non-polluting factor).

We also demonstrate that the ageing of the population (a lower probability to die) reduces the positive influence of the environmental policy on growth, for the aforementioned reasons.

Finally, when an education good is used as input in education activities, the "*generational turnover effect*" continues to operate but the environmental policy enhances growth only if the part of the education good in human capital accumulation remains small. Otherwise, the crowding-out effect leads the environmental policy to be harmful to growth.

Section 2 presents the model. Section 3 is devoted to the study of the existence of the

[4]Following Gradus and Smulders (1993) and many other authors, we model pollution as a flow that originates from production. It corresponds to pollutant emissions like, for example, untimely noise, or non-permanent volatile organic compounds produced by the industry or generated by industrial process (like the use of solvents in consumer and commercial products that corresponds to 28% of the total emissions of VOCs in Canada for 2000). Like (Hettich, 1998, p.292), one should make observe that the "flow" assumption (rather than the "stock" assumption) does not modify the qualitative results along the Balanced Growth Path.

Balanced Growth Path (BGP) equilibrium. Section 4 investigates the influence of the environmental taxation along the BGP and during the transition. Finally, section 5 concludes.

2 The model

Let's consider a Yaari (1965)-Blanchard (1985) overlapping generations model with human capital accumulation and environmental concerns. Time is continuous. Each individual born at time s faces a constant probability of death per unit of time $\lambda \geq 0$. Consequently his life expectancy is $1/\lambda$. When λ increases, the life span decreases. At time s, a cohort of size λ is born. At time $t \geq s$, this cohort has a size equal to $\lambda e^{-\lambda(t-s)}$ and the constant population is equal to $\int_{-\infty}^{t} \lambda e^{-\lambda(t-s)} ds = 1$. There are insurance companies and there is no bequest motive.[5]

The expected utility function of an agent born at $s \leq t$ is:[6]

$$\int_{s}^{\infty} \left[\log c(s,t) - \frac{\zeta}{1+\psi} \mathcal{P}(t)^{1+\psi} \right] e^{-(\varrho+\lambda)(t-s)} dt \tag{1}$$

where $c(s,t)$ denotes consumption in period t of an agent born at time s, $\varrho \geq 0$ is the rate of time preference, $\zeta > 0$ measures the environmental care, that is the weight in utility attached to the environment (capured here by the net flow of pollution $\mathcal{P}$)[7] and $\psi > 0$.

At time t, each agent born at $s \leq t$ can increase his stock of human capital $h(s,t)$ by devoting time to schooling, according to Lucas (1988), and by purchasing $z(s,t)$ units of an educational input, such as

$$\dot{h}(s,t) = B \left[(1 - u(s,t)) h(s,t) \right]^{1-\delta} z(s,t)^{\delta} \tag{2}$$

where $\dot{h}(s,t) \equiv \partial h(s,t)/\partial t$ and $\delta \in [0, 1[$.[8] Parameter B is the efficiency of schooling activities,

[5] The first assumption is made because here death may be interpreted as the termination of a family dynasty and therefore adults who die do not care about what occurs beyond their death. The second assumption is made to avoid unintented bequests.

[6] We use logarithmic utility for the sake of simplicity. Our results remain valid when the intertemporal elasticity of substitution of the consumption is different from unity (see Pautrel, 2008).

[7] Here, we investigate neither the welfare impact of the environmental policy nor the optimal environmental tax. Nevertheless, the net flow of pollution enters as an argument of the utility function because the damages from pollution negatively affect the agents' well-being. See Pautrel (2008) for the welfare analysis and the study of the optimal environmental tax.

[8] When $\delta = z(s,t) = 0$, we obtain the Lucas (1988) human capital accumulation.

$u(s,t) \in]0,1[$ is the part of human capital owned by an agent born at $s \leq t$ that is allocated to productive activities at time t. Note that we make no assumption about the influence of pollution on individual human capital accumulation. Furthermore, we assume that the educational input $z(s,t)$ is produced with final output (one to one).

Households face the following budget constraint:

$$\dot{a}(s,t) = [r(t) + \lambda]\, a(s,t) + u(s,t)h(s,t)w(t) - c(s,t) - z(s,t) \tag{3}$$

where $a(s,t)$ stands for real financial assets in period t and $\omega(t)$ represents the wage rate per effective unit of human capital $u(s,t)h(s,t)$. In addition to the budget constraint, there exists a transversality condition which must be satisfied to prevent households from accumulating debt indefinitely:

$$\lim_{v \to \infty} \left[a_{s,v} e^{-(r+\lambda)(v-t)}\right] = 0$$

The representative agent chooses the time path for $c(s,t)$, his working time $u(s,t)$ and the amount of educational good $z(s,t)$ by maximizing (1) subject to (2) and (3). It yields

$$\dot{c}(s,t) = (r(t) - \varrho)c(s,t) \tag{4}$$

Integrating (3) and (4) and combining the results gives the consumption at time t of an agent born at time s:

$$c(s,t) = (\varrho + \lambda)\left[a(s,t) + \omega(s,t)\right]$$

where $\omega(s,t) \equiv \int_t^\infty \left[u(s,\nu)h(s,\nu)w(\nu)\right] e^{-\int_t^\nu [r(\zeta)+\lambda]d\zeta} d\nu$ is the present value of lifetime earning.

Utility maximization also implies that $u(s,t)$ and the ratio $\frac{z(s,t)}{h(s,t)}$ are independent from s (see appendix A). Conveniently, noting $\tilde{z}(t) \equiv \frac{z(s,t)}{(1-u(s,t))h(s,t)}$, it comes:

$$\tilde{z}(t) = \frac{\delta}{1-\delta} w(t) \tag{5}$$

Finally, utility maximization leads to the equality between the rate of return to human capital and the effective rate of interest:[9]

$$(1-\delta)\frac{\dot{w}(t)}{w(t)} + B(1-\delta)^{1-\delta}\delta^{\delta}w(t)^{\delta} = r(t) + \lambda \tag{6}$$

Due to the simple demographic structure, all individual variables are additive across individuals. Consequently, the aggregate consumption equals

$$C(t) = \int_{-\infty}^{t} c(s,t)\lambda e^{-\lambda(t-s)}ds = (\varrho + \lambda)\left[K(t) + \Omega(t)\right] \tag{7}$$

where $\Omega(t) \equiv \int_{-\infty}^{t} \omega(s,t)\lambda e^{-\lambda(t-s)}ds$ is aggregate human wealth in the economy. The aggregate stock of physical capital is defined by

$$K(t) = \int_{-\infty}^{t} a(s,t)\lambda e^{-\lambda(t-s)}ds,$$

the amount of final output used as educational good is

$$Z(t) \equiv \int_{-\infty}^{t} z(s,t)\lambda e^{-\lambda(t-s)}ds = \tilde{z}(t)(1-u(t))H(t)$$

and the aggregate human capital is

$$H(t) = \int_{-\infty}^{t} h(s,t)\lambda e^{-\lambda(t-s)}ds, \tag{8}$$

We assume that the human capital of an agent born at current date, $h(t,t)$, is inherited from the dying generation. Because the mechanism of intergenerational transmission of knowledge is complex, we make the simplifying assumption that the human capital inherited from the dying generation is a constant part of the aggregate level of human capital such that $h(t,t) = \eta H(t)$ with $\eta \in]0,1]$ (see Song, 2002).[10]

[9] The effective interest rate is the interest rate on the debt r plus the insurance premium λ the agent has to pay when borrowing (Blanchard and Fisher, 1989).

[10] We assume that η could be equal to unity, that is the total aggregate level of human capital is inherited from the dying generation. Because population is constant and normalized to unity, this assumption could be viewed alternatively as the fact that the mean (or per capita) aggregate level of human capital is inherited from the dying generation by each newborn.

The productive sector is competitive. The representative firm produces the final good Y with the following technology:

$$Y(t) = K(t)^{\alpha} \left[\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)} ds \right]^{1-\alpha}, \qquad 0 < \alpha < 1$$

with $Y(t)$ being the aggregate final output. $\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)} ds$ is the amount of the aggregate stock of human capital used in production.

Following Gradus and Smulders (1993), pollution flow is assumed to increase with the stock of physical capital K and reduces with private abatement activities D (made by the firms and which consumes final output one for one):

$$\mathcal{P}(t) = \left[\frac{K(t)}{D(t)} \right]^{\gamma}, \qquad \gamma > 0 \tag{9}$$

We assume that the government implements an environmental policy to incite firms to reduce their net flow of pollution. To do so, the government taxes the net flow of pollution by the firms and transfers to them the fruit of the taxes to fund their abatement activities. Consequently, the representative firm under perfect competition pays a pollution tax on its net pollution $\mathcal{P}(t)$ and it chooses its abatement activities $D(t)$ (whose cost equals $D(t)$) and the amount of factors which maximize its profits $\pi(t) = Y(t) - r(t)K(t) - w(t)\left[\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)} ds\right] - \vartheta(t)\mathcal{P}(t) - D(t) + T(t)^{p}$ where $\vartheta(t)$ is the pollution tax rate and $T^{p}(t)$ denotes transfers from the public sector with $T^{p}(t) = \vartheta(t)\mathcal{P}(t)$. The representative firm takes as given these transfers and pays each production factor at its marginal productivity to maximize profit:

$$r(t) = \alpha \frac{Y(t)}{K(t)} - \vartheta(t)\gamma \frac{\mathcal{P}(t)}{K(t)}$$

$$w(t) = (1-\alpha)K(t)^{\alpha} \left[\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)} ds \right]^{-\alpha} \tag{10}$$

$$D(t) = \vartheta(t)\gamma\mathcal{P}(t) \tag{11}$$

From equations (9) and (11), we obtain:

$$\mathcal{P}(t) = \left[\gamma \frac{\vartheta(t)}{K(t)} \right]^{-\gamma/(1+\gamma)} \tag{12}$$

In the long-run, the net flow of pollution will be constant because K and D will evolve at the same growth rate (see section 3 below). As a result, from equation (12), the environmental tax must rise over time because the physical capital stock accumulates over time (see Hettich, 1998). Intuitively, $\vartheta(t)$ increases over time to encourage firms to increase abatement activities to limit pollution which rises with the physical capital stock. Consequently, we define $\tau \equiv \vartheta(t)/K(t)$, the environmental tax normalized by the physical capital stock and we obtain:

$$\mathcal{P} = \Phi(\tau)^{-\gamma}$$

$$D(t) = \Phi(\tau)K(t)$$

with $\Phi(\tau) \equiv (\gamma\tau)^{1/(1+\gamma)}$. Because τ is fixed by the government and therefore has no transitional dynamics, $\mathcal{P}$ is independent of time.

3 The general equilibrium and the balanced growth path

The final market clearing condition is:

$$Y(t) = C(t) + \dot{K}(t) + D(t) + Z(t),$$

where $Z(t) = \Delta\left(\frac{1-u(t)}{u(t)}\right)Y(t)$ from (5) and (10), with $\Delta \equiv \frac{\delta(1-\alpha)}{1-\delta}$.

Differentiating (8) with respect to time and using the fact that $u(s,t) = u(t)$, the aggregate accumulation of human capital is:

$$\dot{H}(t) = B(1 - u(t))\tilde{z}(t)^{\delta}H(t) - (1 - \eta)\lambda H(t) \tag{13}$$

The first term in the right-hand side of the equation represents the increase in the aggregate human capital due to the investment of each alive generation in education at time t. The second term represents the loss of human capital due to the vanishing of dying generation net from the intergenerational transmission of human capital. Indeed, on the one hand, a part λ of the living cohort born at s with a stock of human capital equal to $h(s,t)\lambda e^{-\lambda(t-s)}$ vanishes reducing growth by $\lambda\int_{-\infty}^{t} h(s,t)\lambda e^{-\lambda(t-s)}ds = \lambda H(t)$ when all generations are aggregated. On the other hand, at the same time, a new cohort of size λ appears, adding $\lambda h(t,t)$ to growth, with $h(t,t) = \eta H(t)$ and $\eta \in]0,1]$ (see above). This net loss reduces the aggregate accumulation of human capital.

Differentiating (7) with respect to time gives

$$\frac{\dot{C}(t)}{C(t)} = \frac{\dot{c}(s,t)}{c(s,t)} - \frac{1}{C(t)} \left[\lambda C(t) - \lambda c(t,t)\right] \tag{14}$$

Aggregate consumption growth differs from individual consumption growth by the term into brackets $-[\lambda C(t) - \lambda c(t,t)]$ which represents what Heijdra and Ligthart (2000) called the "*generational turnover effect*". This effect appears because at each date a cross-section of the existing population dies (reducing aggregate consumption growth by $\lambda C(t)$) and a new generation is born (adding $\lambda c(t,t)$). Because new agents born without financial assets, their consumption $c(t,t)$ is lower than the average consumption $C(t)$ and therefore the "*generational turnover effect*" reduces the growth rate of the aggregate consumption.

Using the expression of $dK(t)/dt$, $d\Omega(t)/dt$ and equation (4) we obtain:

$$\dot{C}(t)/C(t) = r(t) - \varrho - (1-\eta)\lambda - \eta\lambda(\varrho + \lambda)K(t)/C(t) \tag{15}$$

The *generational effect* rises with the probability to die λ: on one hand, agents die at a higher frequency (that increases the generational turnover) and on the other hand the propensity to consume out of wealth $\varrho + \lambda$ increases due to the shorter horizon. Compared with the case where there is no human capital accumulation, a new term η appears that captures the fact that newborns inherit from the dying generation only a part $\eta \in]0,1]$ of the aggregate human-wealth and not the total amount (like in the Yaari (1965)-Blanchard (1985) model).

Using previous results, we can write the dynamics of the model as (see Appendix A):

$$\dot{x}(t) = \left\{\left[\alpha - \left(1 + \Delta\left(1 - \frac{1}{u(t)}\right)\right)\right](b(t)u(t))^{1-\alpha} - \varrho - (1-\eta)\lambda - \eta\lambda(\varrho+\lambda)x(t)^{-1} + x(t)\right\}x(t)$$

$$\dot{b}(t) = \Big\{(1-u(t))B\Delta^{\delta}(b(t)u(t))^{-\alpha\delta} - (1-\eta)\lambda + x(t) + \Phi(\tau) - \left(1 + \Delta\left(1 - \frac{1}{u(t)}\right)\right)(b(t)u(t))^{1-\alpha}\Big\}\, b(t)$$

$$\dot{u}(t) = \Big\{[\alpha^{-1} - 1 + u(t)]\, B\Delta^{\delta}(b(t)u(t))^{-\alpha\delta} + ((\alpha(1-\delta))^{-1} - 1)\,\Phi(\tau) + \Delta\left[\left(1 - \frac{1}{u(t)}\right) - \frac{1}{1-\alpha}\right](b(t)u(t))^{1-\alpha} - ((\alpha(1-\delta))^{-1} + \eta - 1)\,\lambda - x(t)\Big\}\, u(t)$$

where $x(t) \equiv C(t)/K(t)$ and $b(t) \equiv H(t)/K(t)$.

Along the balanced growth path (BGP), C, K, H, D and Y evolve at a common positive rate of growth (denoted $g^\star$, where a * means "*along the BGP*") and the allocation of human capital across sectors is constant. As a consequence, along the balanced growth path $\dot{x} = \dot{b} = \dot{u} = 0$, $x = x^\star$, $b = b^\star$, $u = u^\star$ and $g^\star > 0$.

From the last equation of the dynamical system, we obtain along the BGP:

$$(1-\delta)B\Delta^{\delta}(b^\star u^\star)^{-\alpha\delta} = \alpha(b^\star u^\star)^{1-\alpha} - \Phi(\tau) + \lambda \tag{16}$$

where the left-hand side is the returns to human capital accumulation along the BGP and the right-hand side is the effective interest rate (the returns to physical capital accumulation), also evaluated along the BGP. This relation states $b^\star u^\star$ as an increasing function of τ (because $d\Phi(\tau)/d\tau > 0$), denoted by $\mathcal{R}(B,\tau)$ (with $\partial\mathcal{R}(B,\tau)/\partial\tau > 0$ and $\partial\mathcal{R}(B,\tau)/\partial B > 0$). The two remaining equations of the dynamical system evaluated at the steady-state ($\dot{x}(t) = 0$ and $\dot{b}(t) = 0$) enable us to write the following proposition.

Proposition 1. *There exists a unique* $u^\star \in \,]\underline{u}_\delta, \bar{u}_\delta[$ *with* ($\underline{u}_\delta \equiv \delta + \frac{\varrho+\lambda}{B\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta}}$, $\bar{u}_\delta \equiv 1 - \frac{(1-\eta)\lambda}{B\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta}}$, *and* $0 < \underline{u}_\delta < \bar{u}_\delta < 1$), *solving* $\Gamma_\delta(u;\tau) = 0$ *where* $\Gamma_\delta(u;\tau)$ *is defined as follows*

$$\Gamma_\delta(u;\tau) \equiv \left[(u-\delta)B\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta} - \lambda - \varrho\right] \times \left\{(u-\delta)B\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta} - \eta\lambda + \frac{(1-\alpha)(u^\star-\delta)}{(1-\delta)u}\mathcal{R}(B,\tau)^{1-\alpha}\right\} - \eta\lambda(\varrho+\lambda)$$

with $\mathcal{R}(B,\tau) \equiv b^\star u^\star$ *solution of equation* (16).

Proof. See Appendix A ■

The condition $u^\star < \bar{u}_\delta$ means that the BGP rate of growth must be positive and the condition $u^\star > \underline{u}_\delta$ means that it can not exceed the maximum feasible rate of growth. Such a conditions are conventional in the Lucas (1988) human capital accumulation model.[11]

Finally, the rate of growth along the BGP is:

$$g^\star = B(1-u^\star)\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta} - (1-\eta)\lambda. \tag{17}$$

where $u^\star$ is defined in Proposition 1.

Proposition 2. *The BGP equilibrium is saddle-path stable.*

Proof. See Appendix B. ■

4 The environmental policy and growth

In this section, we investigate the impact of the environmental tax rate τ on the rate of growth, both along the BGP and during the transition. Because the derivation of this influence is analytically cumbersome when $\delta > 0$, we first analytically investigate the case $\delta = z(s,t) = 0$, and then we broaden the analysis to the case $\delta > 0$ using numerical simulations. In both cases, we examine the transitional path of the economy towards the BGP and the transitional influence of the environmental tax.

When $\delta = z(s,t) = 0$, human capital accumulation only depends on the time allocated to education, à la Lucas (1988). In such a case $\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta} = 1$ and from Proposition 1 (with $\delta = 0$), $u^\star$ is the solution of the following equation

$$[Bu^\star - \lambda - \varrho] \times [(\mathcal{A}_0 + u^\star)B + \mathcal{A}_0\Phi(\tau) - (\mathcal{A}_0 + \eta)\lambda] - \eta\lambda(\varrho + \lambda) = 0 \tag{18}$$

with $\mathcal{A}_0 \equiv \alpha^{-1} - 1 > 0$.

[11] See appendix A for details and the textbook by Barro and Sala-i Martin (1995) for examples.

Proposition 3. *$u^\star$ defined by equation* (18) *is a decreasing function of τ and an increasing function of λ, denoted by $\mathcal{U}(\tau,\lambda)$ (with $\partial\mathcal{U}(\tau,\lambda)/\partial\tau < 0$ and $\partial\mathcal{U}(\tau,\lambda)/\partial\lambda > 0$).*

Proof. See Appendix A1. ■

Thus, in the Lucas (1988)'s model with finite lifetime, the allocation of human capital into the production sector along the BGP $u^\star$ falls when the environmental taxation is higher (see the explanations below after Table 2). With $\delta = 0$, the human capital to physical capital ratio along the BGP given by equation (16) is:

$$b^\star = \left[\alpha^{-1}\left(B - \lambda + \Phi(\tau)\right)\right]^{1/(1-\alpha)} \mathcal{U}(\tau,\lambda)^{-1} > 0$$

A tighter environmental tax leads the final production to be less intensive in physical capital and more intensive in human capital because it increases the cost of physical capital: $b^\star$ rises. The aggregate consumption to physical capital ratio along the BGP is given by:

$$x^\star = \frac{\lambda\eta(\varrho + \lambda)}{B\,\mathcal{U}(\tau,\lambda) - \lambda - \varrho} > 0$$

and increases with the environmental tax rate. Finally, the growth rate along the BGP, defined by equation (17), becomes with $\delta = 0$:

$$g^\star = B(1 - \mathcal{U}(\tau,\lambda)) - (1-\eta)\lambda \tag{19}$$

Proposition 4. *The environmental policy has a positive impact on the Balanced growth path rate of growth.*

Proof. It comes directly from Proposition 3 and equation (19). ■

Consequently, when lifetime is finite it is possible to implement a win-win environmental policy in a Lucas (1988) growth model. Furthermore, when the horizon extends (λ decreases), the allocation of human capital into final production $u^\star$ drops: agents invest more in human capital. And when lifetime is infinite ($\lambda = 0$), the allocation of human capital into the production $u^\star$ is independent from τ along the balanced growth path (as demonstrated in Appendix A1).

Proposition 5. *The ageing of the population (a lower λ) reduces the influence of the environmental policy on the BGP rate of growth. When lifetime is infinite, the BGP rate of growth is not affected by the environmental policy.*

Proof. See Appendix A1. ■

We also investigate the trajectory of the economy out of the steady-state and the influence of the environmental tax during this transition. Due to the complexity of the model, we use the Time-Elimination Method to perform the numerical analysis.[12] We calibrate the model to obtain realistic values of the growth rate of GDP and the probability of death for the US economy. From the *World Development Indicators 2005* by the World Bank, life expectancy was 77.4 years in 2003, the growth rate was 3.3% during the period 1990-2002 and a public health expenditures as percentage of GDP was 6.55%. Since the expected lifetime is the reverse of the probability of death per unit of time λ, we want λ to be close to $1/77.4 = 0.0128$. We adjust other variables to obtain such values for our benchmark case.

Table 1 summarizes the benchmark value of parameters and Table 2 summarizes the exercise of comparative statics. Graph 1 and Graph 2 draw the temporal evolution of the main variables towards the new steady-state when an unanticipated increase in the environmental tax is implemented by the government, respectively for infinite and finite lifetime.

Table 1. Benchmark value of parameters

α	η	ϱ	B	γ	$\lambda^{(1)}$	$\lambda^{(2)}$
0.3	0.85	0.025	0.075	0.3	0.0128	0

$^{(1)}$ finite lifetime $^{(2)}$ infinite lifetime

Table 2. The increase in the environmental tax along the BGP

[12]See Mulligan and Sala-i Martin (1991) for details about the Time-Elimination Method, and Mulligan and Sala-i Martin (1993) for an application of the two-sector models of endogenous growth.

	Finite lifetime				*Infinite lifetime*	
	$\lambda = 0.0128$		$\lambda = 0.0200$			
	$\tau = 0.01$	$\tau = 0.1$	$\tau = 0.01$	$\tau = 0.1$	$\tau = 0.01$	$\tau = 0.1$
$g^\star$	3.32%	3.40%	2.29%	2.45%	5%	5%
$\mathcal{P}^\star$	3.82	2.25	3.82	2.25	3.82	2.25
$u^\star$	0.5313	0.5206	0.6545	0.6323	1/3	1/3
$r^{\star(1)} + \lambda$	0.075	0.075	0.075	0.075	0.075	0.075
$x^{\star(2)}$	0.2009	0.3305	0.1872	0.3160	0.2268	0.3572
$b^{\star(3)}$	0.2532	0.5790	0.1774	0.4394	0.5073	1.0345

$^{(1)}\alpha Y/K - \Phi(\tau)$ $^{(2)}C/K$ $^{(3)}H/K$

To understand the economic mechanisms underlying Propositions 3, 4 and 5, we first consider the case where lifetime is infinite ($\lambda = 0$), there is a single representative household and the "*generational turnover effect*" is absent. In such a case, the environmental policy, through a tighter environmental tax, has two effects. First, a crowding-out effect, due to the rise of abatement expenditures, reduces consumption and investment. Second, a factorial reallocation effect leads to a production more intensive in human capital: there is a substitution of the pollutant factor (physical capital) by the "clean" factor (human capital). The two effects offset to keep the interest rate constant and the growth rate along the BGP unchanged (see Gradus and Smulders, 1993).

When agents have finite lives ($\lambda > 0$), the tighter environmental policy has a third impact: a "*generational turnover effect*" that affects the aggregate consumption rate of growth. Indeed, with finite lifetime, the aggregate consumption rate of growth differs from the individual consumption rate of growth $r - \varrho$, by the "*proportionnal* " difference between average consumption and consumption by newly born households $[C(t) - c(t,t)]/C(t)$ (see equation 14). And that "*generational turnover*" explains the positive impact of the environnmental taxation of the BGP growth with finite lifetime.

Indeed, at the impact, the tighter environmental tax brings down the returns to capital below the returns to education (see equation (16) with $\delta = 0$) both in infinite and finite lifetime, as shown respectively by Graph 1*iv* and Graph 2*iv*. Agents allocate a part of their resources from final output to human capital accumulation: u jumps downward and the wage rises (see equation (10), Graphs 1*i-v* and Graphs 2*i-v*). The discounted value of earnings (that is the human wealth) increases and the interest payments on non-human wealth (K is given because it is pre-determined) fall: agents reduce their saving and consumption jumps at impact (see the transitional dynamics of the aggregate consumption to physical capital ratio $x(t) \equiv C(t)/K(t)$ in Graph 1*iii* and Graph 2*iii*). Because with finite lifetime the horizon is shorter and at each date a new generation born without financial assets, that jump in aggregate consumption $C(t)$ is lower compared with the infinite lifetime case, as shown in Graph 1*ix* and Graph 2*ix*.[13] In addition, the growth rate of the aggregate consumption to physical capital ratio $\dot{x}(t)/x(t)$ jumps upward at impact because the fall in aggregate physical capital growth at impact is higher than the fall in aggregate consumption growth. During the transition, $\dot{x}(t)/x(t)$ decreases towards 0 while aggregate physical capital growth and aggregate consumption growth converge towards their new BGP equilibrium (see Graphs 1*vii-ix* and Graphs 2*vii-ix*). After the tax shock, the aggregate consumption to physical capital ratio x is always higher than its value before the tightening of the environmental taxation (see Graph 1*iii* and Graph 2*iii*). Because aggregate consumption growth with finite lifetime is negatively affected by the generational turnover term $[C(t) - c(t,t)]/C(t) = K(t)/C(t) = x(t)^{-1}$ (see equations 14 and 15), it is always above its initial value, for a given level of interest rate. Therefore, when aggregate consumption growth goes back to its initial level, the interest rate is lower than its initial value (that is lower than the returns to education).[14] Thus, the substitution between physical capital and human capital continues such that at the new BGP equilibrium (when the interest rate backs to its initial value) $u^\star$ remains lower than its value before the tightening of the environmental tax, and human capital accumulation, the aggregate consumption and the physical capital rates of growth are higher.[15]

[13]That explanation is based on the one given by Blanchard and Fisher (1989, p.138-139) who consider the impact of an increase in the interest rate in a continuous time overlapping generations model with no endogenous growth.

[14]Using our numerical simulation for the finite lifetime case, we find that $\dot{C}(t)/C(t) = 3.32\%$ (its initial level reported in Table 2) at $t = 18$ while at that time $r(t) + \lambda = 0.0742$ is lower than its initial level 0.075 (see Table 2). Note that $r(t) + \lambda$ goes back to its initial level 0.075 at $t = 50$.

[15]Note that when pollutant emissions enter the final output production function as a factor, the environmental

The longer the horizon (the lower λ), the lower the influence of the *"generational turnover"* and the lower the fall in u to go the the new BGP equilibrium. For a given rise of the environmental tax, the allocation of human capital to the output sector along the BGP ($u^{\star}$) reduces less for a lower λ than for a higher λ.

In the remaining of the section, we broaden our analysis to the case $\delta > 0$, that is an educational good is used in education besides time. Conversely to the case $\delta = 0$ we dealt with, the influence of the environmental taxation on the allocation of human capital into the manufacturing sector $u^{\star}$ is not clear-cut when lifetime is finite. To understand the reason, we back to the infinite lifetime case. As demonstrated in Appendix A2, in that case, $u^{\star}$ rises with τ because the tighter environmental policy not only crowds-out consumption and physical capital accumulation, but also the part of output allocated to the education sector Z. As a result, the rewards to education falls below its initial value and the agents reallocate their time to production to compensate the decrease in their consumption. The BGP rate of growth falls and the environmental tax is detrimental to growth. When lifetime is finite, the aforementioned effect exists besides the "*generational turnover effect*" that operates in the opposite way. To investigate whether the "*generational turnover effect*" is high enough to compensate or to offset the crowding-out effect, we use numerical simulations. In particular, we examine the increase in environmental taxation for different values of δ, insofar as we demonstrated that the environmental policy enhances growth when only time is used as input for education (that is $\delta = 0$) and because we expect that environmental policy is harmful for growth when only final output is used to increase human capital (that is $\delta = 1$).[16]

Table 3. Impact of the environmental policy according to δ.

taxation does not affect growth in the long-run despite finite lifetime (see Pautrel (2008) for a formal proof).

[16] In such a case, the technology to accumulate human capital is similar to the technology of final output production and physical capital accumulation except the parameter B (see equation 2).

	$\delta = 0.01$		$\delta = 0.06$		$\delta = 0.1$		$\delta = 0.5$	
	$\tau = 0.01$	$\tau = 0.1$	$\tau = 0.01$	$\tau = 0.1$	$\tau = 0.01$	$\tau = 0.1$	$\tau = 0.01$	$\tau = 0.065$
$g^\star$	2.92%	2.99%	1.8766%	1.8821%	1.3825%	1.3378%	0.4071%	0.0205%
$\mathcal{P}^\star$	3.82	2.25	3.82	2.25	3.82	2.25	3.82	2.48
$u^\star$	0.5663	0.5556	0.6812	0.6750	0.7480	0.7478	0.9362	0.9747
$r^\star + \lambda$	0.0711	0.0709	0.0610	0.0600	0.0562	0.0546	0.0470	0.0419
$x^\star$ (C/K)	0.19061	0.3187	0.1645	0.2874	0.1530	0.2736	0.1295	0.2051
$b^\star$ (H/K)	0.2198	0.5183	0.1461	0.3745	0.1181	0.3156	0.0725	0.1484

Positive effect on growth (δ = 0.01, δ = 0.06) — *Negative effect on growth* (δ = 0.1, δ = 0.5)

Results of the numerical simulations are reported in table 3.[17] They show that, for the parameter values chosen, Proposition 4 still holds when an education good (produced with human and physical capital) is introduced in the technology of education, only if the part of this education good in human capital accumulation remains small enough. Indeed, when the relative part of this education good in the human capital accumulation becomes important, the crowding-out effect offsets the positive effect arising from the "*generational turnover effect*" . The BGP rate of growth drops.

5 Conclusion

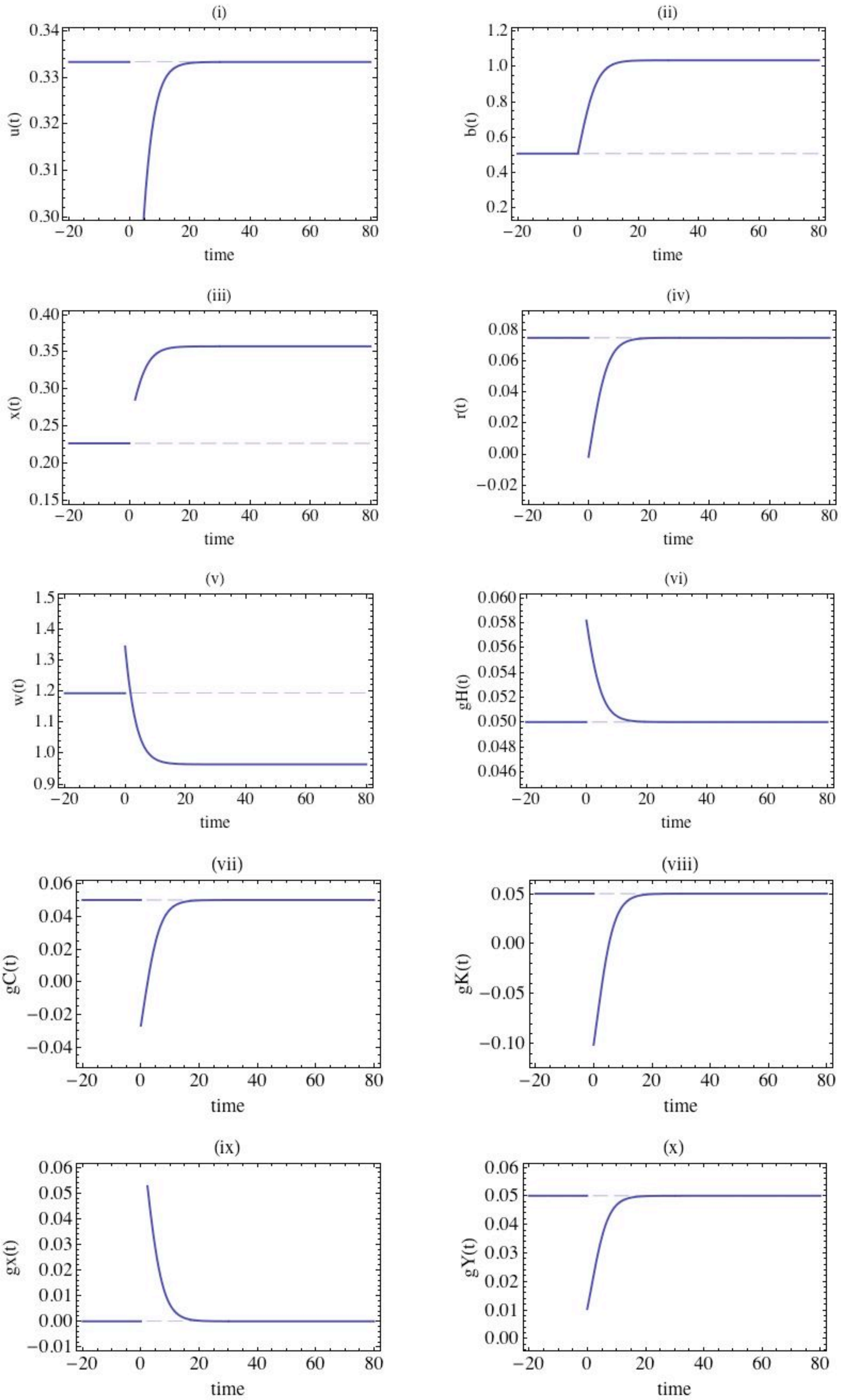
In this article, we demonstrate that, if finite lifetime is taken into account, a win-win environmental policy may be implemented in an economy where growth is driven by human capital accumulation à la Lucas (1988) and the source of pollution is physical capital, while pollution does not influence educational activities, labor supply is not elastic and human capital does not enter the utility function. This is because finite lifetime and the appearance of newborns at each date creates a turnover of generations which disconnects the aggregate consumption growth to the after-tax interest rate. We also demonstrate that the ageing of the population (a lower probability to die) reduces the positive influence of the environmental policy on growth.

Finally, using numerical simulations, we show that the positive impact of a tighter environmental tax on the growth rate still holds when an education good (produced with human and

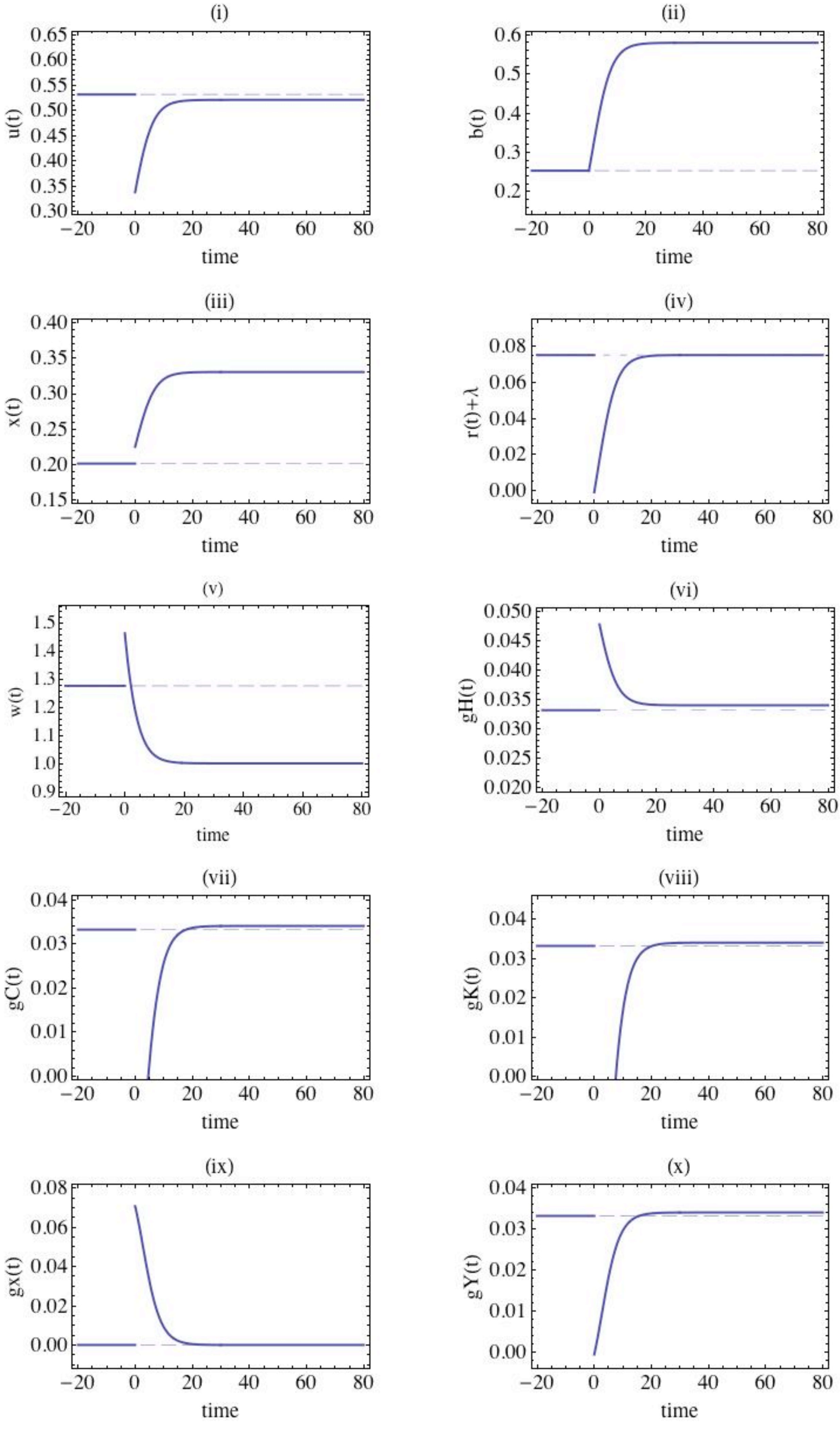
[17]Note that in our numerical simulations, when $\delta = 0.5$, $u^\star \notin]\underline{u}_\delta, \bar{u}_\delta[$ and $g^\star < 0$ when $\tau > 0.065$.

physical capital) is introduced in the technology of education, only if the part of this education good in human capital accumulation is small enough to prevent the crowding-out effect of the tax to offset the "*generational turnover effect*" arising from finite lifetime.

Graph 1. Infinite lifetime



Graph 2. Finite lifetime



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Remark: In the following appendix, references to equations without the prefix A are references to the equations in the body of the paper.

Appendix A

In this appendix, we first solve the general model of Section 2, where $\delta \geq 0$. Then, we focus our attention to the case $\delta = 0$ (Appendix A1). Finally, we investigate the case $\delta > 0$ and $\lambda = 0$ (Appendix A2).

The program of the households is:

$$\begin{aligned} \max_{c(s,t),z(s,t),a(s,t),h(s,t),u(s,t)} \quad & \textstyle\int_s^\infty \left[\log c(s,t) - \dfrac{\zeta}{1+\psi}\mathcal{P}(t)^{1+\psi}\right] e^{-(\varrho+\lambda)(t-s)}dt \\ s.t. \quad & \dot{a}(s,t) = \left[r(t)+\lambda\right] a(s,t) + u(s,t)h(s,t)w(t) - c(s,t) - z(s,t) \\ & \dot{h}(s,t) = B(1-u(s,t))^{1-\delta} h(s,t)^{1-\delta} z(s,t)^{\delta} \\ & a(s,s,) = 0 \qquad h(s,s) = \eta H(s) > 0 \end{aligned}$$

The Hamiltonian of the program may be written as:

$$\begin{aligned} \mathcal{H} = \left[\log c(s,t) - \frac{\zeta}{1+\psi}\mathcal{P}(t)^{1+\psi}\right] + \pi_1(t)\left[(r(t)+\lambda)a(s,t) + u(s,t)h(s,t)w(t) - c(s,t) - z(s,t)\right] \\ + \pi_2(t)B(1-u(s,t))^{1-\delta}h(s,t)^{1-\delta}z(s,t)^{\delta} \end{aligned}$$

The F.O.C. are

$$\frac{\partial \mathcal{H}}{\partial c(s,t)} = 0 \quad \Rightarrow \quad \frac{1}{c(s,t)} = \pi_1(t) \tag{A.1}$$

$$\frac{\partial \mathcal{H}}{\partial z(s,t)} = 0 \quad \Rightarrow \quad \pi_1(t) = \pi_2(t)\delta B(1-u(s,t))^{1-\delta}\left(\frac{z(s,t)}{h(s,t)}\right)^{\delta-1} \tag{A.2}$$

$$\frac{\partial \mathcal{H}}{\partial u(s,t)} = 0 \quad \Rightarrow \quad \pi_1(t)w(t) = \pi_2(t)B(1-\delta)(1-u(s,t))^{-\delta}\left(\frac{z(s,t)}{h(s,t)}\right)^{\delta} \tag{A.3}$$

$$\frac{\partial \mathcal{H}}{\partial a(s,t)} = -\dot{\pi}_1(t) + (\varrho+\lambda)\pi_1(t) \quad \Rightarrow \quad \pi_1(t)(r(t)+\lambda) = -\dot{\pi}_1(t) + (\varrho+\lambda)\pi_1(t) \tag{A.4}$$

$$\frac{\partial \mathcal{H}}{\partial h(s,t)} = -\dot{\pi}_2(t) + (\varrho+\lambda)\pi_2(t) \quad \Rightarrow$$

$$\pi_1(t)w(t)u(s,t) + \pi_2(t)B(1-\delta)(1-u(s,t))^{1-\delta}\left(\frac{z(s,t)}{h(s,t)}\right)^{\delta} = -\dot{\pi}_2(t) + (\varrho+\lambda)\pi_2(t) \tag{A.5}$$

Equation (A.3) implies that the ratio $\frac{z(s,t)}{(1-u(s,t))h(s,t)}$ is independent from s and equation (A.5) implies that $u(s,t)$ is independent from s. Consequently, the ratio $\frac{z(s,t)}{h(s,t)}$ is independent from s. Conveniently, we denote $\tilde{z}(t) \equiv \frac{z(s,t)}{(1-u(s,t))h(s,t)}$.

From (A.1) and (A.4), we obtain

$$\dot{c}(s,t) = (r(t) - \varrho)c(s,t) \tag{4}$$

Equations (A.2) and (A.3) give:

$$\tilde{z}(t) = \frac{\delta}{1-\delta} w(t) \tag{5}$$

And from equations (A.2) and (A.5):

$$\frac{\dot{\pi}_2(t)}{\pi_2(t)} = \varrho + \lambda - B\left[1-\delta\right]\tilde{z}(t)^{\delta}$$

Differentiating (A.3) with respect to time, we obtain:

$$\frac{\dot{\pi}_1(t)}{\pi_1(t)} + \frac{\dot{w}(t)}{w(t)} = \frac{\dot{\pi}_2(t)}{\pi_2(t)} + \delta\frac{\dot{\tilde{z}}(t)}{\tilde{z}(t)}$$

Replacing by the expressions of $\frac{\dot{\pi}_1(t)}{\pi_1(t)}$ and $\frac{\dot{\pi}_2(t)}{\pi_2(t)}$, it gives

$$\frac{\dot{w}(t)}{w(t)} - \delta\frac{\dot{\tilde{z}}(t)}{\tilde{z}(t)} + B(1-\delta)\tilde{z}(t)^{\delta} = r(t) + \lambda$$

which means that the returns to education must be equal to the returns to physical capital.

We can re-write this relation in terms of $w(t)$ and $r(t)$:

$$(1-\delta)\frac{\dot{w}(t)}{w(t)} + B(1-\delta)^{1-\delta}\delta^{\delta}w(t)^{\delta} = r(t) + \lambda \tag{6}$$

From equations (10) and (5), it is possible to express $\tilde{z}(t)$ in terms of $Y(t)$:

$$\tilde{z}(t) = \frac{\delta}{1-\delta}(1-\alpha)\frac{Y(t)}{u(t)H(t)}$$

Now, we can write that the amount of final output used as educational good is $Z(t) \equiv \int_{-\infty}^{t} z(s,t)\lambda e^{-\lambda(t-s)}ds = \int_{-\infty}^{t}\left(\frac{z(s,t)}{(1-u(s,t))h(s,t)}\right)(1-u(s,t))h(s,t)\lambda e^{-\lambda(t-s)}ds = \tilde{z}(t)(1-u(t))\int_{-\infty}^{t} h(s,t)\lambda e^{-\lambda(t-s)}ds = \tilde{z}(t)(1-u(t))H(t)$. Consequently:

$$Z(t) = \frac{\delta(1-\alpha)}{1-\delta}\left(\frac{1-u(t)}{u(t)}\right)Y(t),$$

the market clearing condition is written as:

$$\left(1 + \frac{(1-\alpha)\delta}{1-\delta}\left(1 - \frac{1}{u(t)}\right)\right)Y(t) = \dot{K}(t) + C(t) + \Phi(\tau)K(t)$$

and the aggregate accumulation of human capital is:

$$\dot{H}(t) = \left[B(1-u(t))\tilde{z}(t)^{\delta} - (1-\eta)\lambda\right] H(t) \tag{13}$$

Finally, from equations (6), (10) and the fact that $\int_{-\infty}^{t} u(s,t)h(s,t)\lambda e^{-\lambda(t-s)}ds = u(t)H(t)$ because $u(s,t) = u(t)$, we obtain

$$\frac{\dot{u}(t)}{u(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{H}(t)}{H(t)} - \alpha^{-1}\left[r(t) + \lambda - B(1-\delta)^{1-\delta}\delta^{\delta}(b(t)u(t))^{-\alpha\delta}\right]$$

Therefore, the dynamical system is summarized by

$$\begin{aligned}
\dot{x}(t) = \;& \left\{\left[\alpha - \left(1 - \Delta\left(\frac{1}{u(t)} - 1\right)\right)\right](b(t)u(t))^{1-\alpha} - \varrho - (1-\eta)\lambda - \eta\lambda(\varrho+\lambda)x(t)^{-1} + x(t)\right\} x(t) \\
\dot{b}(t) = \;& \Big\{(1-u(t))B\Delta^{\delta}(b(t)u(t))^{-\alpha\delta} \\
& -(1-\eta)\lambda + x(t) + \Phi(\tau) - \left(1 - \Delta\left(\frac{1}{u(t)} - 1\right)\right)(b(t)u(t))^{1-\alpha}\Big\} b(t) \\
\dot{u}(t) = \;& \Big\{\left[\alpha^{-1} - 1 + u(t)\right] B\Delta^{\delta}(b(t)u(t))^{-\alpha\delta} + \left((\alpha(1-\delta))^{-1} - 1\right)\Phi(\tau) \\
& -\Delta\left[\left(\frac{1}{u(t)} - 1\right) + \frac{1}{1-\alpha}\right](b(t)u(t))^{1-\alpha} - \left((\alpha(1-\delta))^{-1} + \eta - 1\right)\lambda - x(t)\Big\} u(t)
\end{aligned} \tag{A.6}$$

with $\Delta \equiv \frac{(1-\alpha)\delta}{1-\delta}$.

The two last equations of the dynamical system (A.6) evaluated in the steady-state ($\dot{b}(t) = 0$ and $\dot{u}(t) = 0$) gives $b^{\star}u^{\star}$ as the solution of the following equation

$$(1-\delta)B\Delta^{\delta}(b^{\star}u^{\star})^{-\alpha\delta} = \alpha(b^{\star}u^{\star})^{1-\alpha} - \Phi(\tau) + \lambda \tag{16}$$

where the left-hand side is the returns to human capital accumulation along the BGP and the right-hand side is the effective interest rate (the returns to physical capital accumulation, see equation (6)),[18] both evaluated along the BGP.

When $\delta \in]0,1[$, the left-hand side is a decreasing function of $b^{\star}u^{\star} \in]0,+\infty[$ with $\lim_{b^{\star}u^{\star}\to 0} LHS = +\infty$ and $\lim_{b^{\star}u^{\star}\to +\infty} LHS = 0$, and the right-hand side is an increasing function of $b^{\star}u^{\star}$ with $\lim_{b^{\star}u^{\star}\to 0} RHS = \lambda - \Phi(\tau)$ and $\lim_{b^{\star}u^{\star}\to +\infty} RHS = +\infty$. Consequently, the equation (16) defines a unique $b^{\star}u^{\star} \in]0,+\infty[$. Because $\delta \in]0,1[$ and $d\Phi(\tau)/d\tau > 0$, it is straightforward using the theorem of the implict function that $b^{\star}u^{\star}$ is an increasing function of τ and B. When $\delta = 0$, equation (16) becomes $B = \alpha(b^{\star}u^{\star})^{1-\alpha} - \Phi(\tau) + \lambda$ and defines an explicit expression for $b^{\star}u^{\star}$:

$$b^{\star}u^{\star} = \left(\frac{B - \lambda + \Phi(\tau)}{\alpha}\right)^{1/(1-\alpha)}$$

[18]See footnote 9 page 7, for the definition of the effective interest rate.

$b^\star u^\star$ is always an increasing function of B and τ and is positive under the sufficient condition that the returns to education B corrected by the probability of death λ is positive:

$$B - \lambda > 0 \tag{A.7}$$

For convenience, we denote $\mathcal{R}(B,\tau)$, the solution $b^\star u^\star$ of equation (16), with $d\mathcal{R}(B,\tau)/d\tau > 0$ and $d\mathcal{R}(B,\tau)/dB > 0$ for $\delta \in [0,1[$.

Along the BGP, $\dot{x} = \dot{b} = 0$ defines $x^\star$ as follows:

$$x^\star = \frac{\eta\lambda(\varrho+\lambda)}{\alpha\mathcal{R}(B,\tau)^{1-\alpha} - \Phi(\tau) - \varrho - (1-u^\star)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}} \tag{A.8}$$

Using equation (16) we can re-write (from the previous expression) $x^\star$ as a function $\mathcal{X}_1(u^\star,\tau)$:

$$x^\star = \mathcal{X}_1(u^\star,\tau) \equiv \frac{\eta\lambda(\varrho+\lambda)}{(u^\star-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} - \lambda - \varrho}$$

To obtain $x^\star > 0$, we impose that

$$(u^\star - \delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} > \varrho + \lambda \tag{A.9}$$

ensuring that human capital will not be fully invested in human capital accumulation along the balanced growth path:

$$u^\star > \underline{u}_\delta, \qquad \text{with} \qquad \underline{u}_\delta \equiv \delta + \frac{\varrho+\lambda}{B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}} \in]0,1[\tag{C1}$$

This condition ensures that the growth rate of human capital does not exceed the maximum feasible rate (when the total amount of human capital is allocated to education).

We also assume that the balanced growth path rate of growth $g^\star$ must be positive, that is (from equation 13):

$$(1-u^\star)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} > (1-\eta)\lambda \tag{A.10}$$

This assumption imposes that the investment in education is positive (from equation 13):[19]

$$u^\star < \bar{u}_\delta, \qquad \text{with} \qquad \bar{u}_\delta \equiv 1 - \frac{(1-\eta)\lambda}{B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}} \in]0,1[\tag{C2}$$

Under conditions (C1)-(C2) the following inequality holds (by summing (A.9) and (A.10)):

$$(1-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} > \varrho + (2-\eta)\lambda, \qquad \text{for } \delta \in [0,1[\tag{A.11}$$

and enables us to demonstrate that $\underline{u}_\delta < \bar{u}_\delta$.

[19]Conditions (C1)-(C2) ensure that the BGP growth rate of individual consumption $\dot{c}(s,t)/c(s,t)|_{BGP}$ is higher than the individual accumulation of human capital $\dot{h}(s,t)/h(s,t)|_{BGP}$, that is $c(s,t)|_{BGP} > 0$ (see the denominator of equation A.8).

Because $\eta \in]0,1]$, conditions (C1)-(C2) impose $(1-\delta)B\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta} > \varrho + \lambda > (1-\eta)\lambda$ (from the inequality (A.11)), that is they imply a positive growth rate of individual consumption $c(s,t)$ (see equations 4 and 16). Finally, condition (A.7) is verified under conditions (C1)-(C2) (from the inequality (A.11)). Furthermore $\lim_{u^\star \to \underline{u}_\delta} \chi_1(u^\star;\tau) = +\infty$ and $\lim_{u^\star \to \bar{u}_\delta} \chi_1(u^\star;\tau) = \frac{\eta\lambda(\varrho+\lambda)}{(1-\delta)B\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta}-(2-\eta)\lambda-\varrho} > 0$.

From $\dot{u}(t) = 0$, we can also define $x^\star$ as follows:

$$x^\star = \left[\alpha^{-1} - 1 + u^\star\right] B\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta} + \mathcal{A}_\delta \Phi(\tau) - \Delta\left(\frac{\alpha}{1-\alpha} + \frac{1}{u^\star}\right)\mathcal{R}(B,\tau)^{1-\alpha} - \left(\mathcal{A}_\delta + \eta\right)\lambda \quad \text{(A.12)}$$

with $\Delta \equiv \frac{(1-\alpha)\delta}{1-\delta}$ and $\mathcal{A}_\delta \equiv (\alpha(1-\delta))^{-1} - 1 > 0$, $\forall \delta \in [0,1[$. Using (16) and simplifying, we can express $x^\star$ as a function $\mathcal{X}_2(u^\star,\tau)$:

$$x^\star = \mathcal{X}_2(u^\star,\tau) \equiv (u^\star - \delta)B\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta} - \eta\lambda + \frac{(1-\alpha)(u^\star-\delta)}{(1-\delta)u^\star}\mathcal{R}(B,\tau)^{1-\alpha} \quad \text{(A.13)}$$

It is straightforward that $\chi_2(u^\star;\tau)$ is an increasing function of $u^\star$ and $\chi_2(u^\star;\tau) > 0$ for all $u^\star \in]\underline{u}_\delta, \bar{u}_\delta[$ and $\delta \in [0,1[$. When $\delta = 0$, it is straightforward that $\chi_2(u^\star;\tau)$ is an increasing function of τ, but for $\delta \in]0,1[$, the influence of τ is unclear.

The BGP equilibrium is defined by $\chi_1(u^\star;\tau) = \chi_2(u^\star;\tau)$ for $u^\star \in]\underline{u}_\delta, \bar{u}_\delta[$. That is, there exists, for $\delta \in [0,1[$, a unique $u^\star \in]\underline{u}_\delta, \bar{u}_\delta[$, solution of $\Gamma_\delta(u;\tau) = 0$ with

$$\Gamma_\delta(u;\tau) \equiv \left[(u-\delta)B\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta} - \lambda - \varrho\right] \times \left\{(u-\delta)B\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta} - \eta\lambda + \frac{(1-\alpha)(u^\star-\delta)}{(1-\delta)u}\mathcal{R}(B,\tau)^{1-\alpha}\right\} - \eta\lambda(\varrho+\lambda) \quad \text{(A.14)}$$

and $\mathcal{R}(B,\tau)$ is defined by equation (16).

It is straightforward that, for $u^\star \in]\underline{u}_\delta, \bar{u}_\delta[$ (see conditions (C1)-(C2)), $\Gamma_\delta(\underline{u}_\delta;\tau) = -\eta\lambda(\varrho+\lambda) < 0$ and $\Gamma_\delta(\bar{u}_\delta;\tau) > 0$.[20] Because in the interval $]\underline{u}_\delta, \bar{u}_\delta[$, $\Gamma_\delta(u;\tau)$ is a monotonic increasing function of u, $u^\star$ solution of $\Gamma_\delta(u;\tau) = 0$ is unique. The influence of τ on $u^\star$ is not clear except when $\delta = 0$.

Appendix A1. The case $\delta = 0$

When $\delta = 0$, we have $\Delta = 0$ and $\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta} = 1$. Conditions (C1) and (C2) hold (with $\delta = 0$ and $\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta} = 1$) and there exists a unique $u^\star \in]\underline{u}_0, \bar{u}_0[$ solution of $\Gamma(u;\tau) = 0$ where $\Gamma(u;\tau)$ is defined, from equation (A.14) with $\delta = 0$, as follows:

$$\Gamma(u;\tau) \equiv [Bu - \lambda - \varrho] \times [(\mathcal{A}_0 + u)B + \mathcal{A}_0\Phi(\tau) - (\mathcal{A}_0 + \eta)\lambda] - \eta\lambda(\varrho+\lambda) = 0 \quad (18)$$

[20] Because $\left[(1-\delta)B\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta} - (2-\eta)\lambda - \varrho\right]\left[(1-\delta)B\Delta^{\delta}\mathcal{R}(B,\tau)^{-\alpha\delta} - \lambda\right] > \eta\lambda(\varrho+\lambda)$, we obtain $\Gamma_\delta(\bar{u}_\delta;\tau) > 0$.

with $\Gamma(\underline{u}_0;\tau)=-\eta\lambda(\varrho+\lambda)<0$ and $\Gamma(\bar{u}_0;\tau)=[B-(2-\eta)\lambda-\varrho]\times\left[\alpha^{-1}(B-\lambda)+\mathcal{A}_0\Phi(\tau)\right]-\eta\lambda(\varrho+\lambda)>0$.[21] From the implicit function theorem, the influence of τ on $u^\star$ is given by $u^{\star\prime}=-\frac{\partial\Gamma(u;\tau)/\partial\tau}{\partial\Gamma(u;\tau)/\partial u}$. Noting $\Gamma(u;\tau)=\Gamma_1(u;\tau)\times\Gamma_2(u;\tau)-\eta\lambda(\varrho+\lambda)$, with $\Gamma_1(u;\tau)\equiv Bu^\star-\lambda-\varrho>0$ and $\Gamma_2(u;\tau)\equiv(\mathcal{A}_0+u)B+\mathcal{A}_0\Phi(\tau)-(\mathcal{A}_0+\eta)\lambda>0$, we obtain $u^{\star\prime}=\frac{-\mathcal{A}\Phi'(\tau)\Gamma_1(u;\tau)}{B[\Gamma_1(u;\tau)+\Gamma_2(u;\tau)]}$ is negative and $u^\star$ is a decreasing function of τ.

If we note $g^{\star\prime}\equiv dg^\star/d\tau=-Bu^{\star\prime}$, the effect of the environmental policy on growth with respect to the horizon is given by $dg^{\star\prime}/d\lambda$. Because $g^{\star\prime}=\mathcal{A}\Phi'(\tau)\left[1+\frac{\Gamma_2(u;\tau)}{\Gamma_1(u;\tau)}\right]^{-1}$, and $\frac{\partial u^\star}{\partial\lambda}=\frac{\Gamma_2(u;\tau)+(\mathcal{A}_0+\eta)\Gamma_1(u;\tau)}{B[\Gamma_1(u;\tau)+\Gamma_2(u;\tau)]}>0$, and $\frac{\partial\Gamma_1(u;\tau)}{\partial\lambda}=(\alpha^{-1}-2+\eta)\frac{\Gamma_1(u;\tau)}{\Gamma_1(u;\tau)+\Gamma_2(u;\tau)}>0$ (under the realistic sufficient condition $\alpha<1/(2-\eta)$) and $\frac{\partial\Gamma_2(u;\tau)}{\partial\lambda}=-(\alpha^{-1}-2+\eta)\frac{\Gamma_2(u;\tau)}{\Gamma_1(u;\tau)+\Gamma_2(u;\tau)}<0$, we obtain that $\frac{\partial\Gamma_2(u;\tau)/\Gamma_1(u;\tau)}{\partial\lambda}=-2(\alpha^{-1}-2+\eta)\frac{\Gamma_2(u;\tau)}{\Gamma_1(u;\tau)}<0$ that is $\frac{\partial g^{\star\prime}}{\partial\lambda}>0$.

When $\lambda=0$, equation (18) gives the solution of the Lucas (1988) model with logarithmic utility: $u^\star=\varrho/B$ and $g^\star=B-\varrho$.

Appendix A2. The case $\delta>0$ and $\lambda=0$

When $\delta>0$ and $\lambda=0$, the dynamical system (A.6) becomes:

$$
\begin{aligned}
\dot{x}(t) &= \left\{\left[\alpha-\left(1+\Delta\left(1-\frac{1}{u(t)}\right)\right)\right](b(t)u(t))^{1-\alpha}-\varrho+x(t)\right\}x(t)\\
\dot{b}(t) &= \left\{(1-u(t))B\Delta^\delta(b(t)u(t))^{-\alpha\delta}+x(t)+\Phi(\tau)-\left(1+\Delta\left(1-\frac{1}{u(t)}\right)\right)(b(t)u(t))^{1-\alpha}\right\}b(t)\\
\dot{u}(t) &= \left\{\left[\alpha^{-1}-1+u(t)\right]B\Delta^\delta(b(t)u(t))^{-\alpha\delta}+\left((\alpha(1-\delta))^{-1}-1\right)\Phi(\tau)-x(t)\right.\\
&\quad\left.+\Delta\left[\left(1-\frac{1}{u(t)}\right)-\frac{1}{1-\alpha}\right](b(t)u(t))^{1-\alpha}\right\}u(t)
\end{aligned}
$$

From $\dot{u}=0$ and $\dot{b}=0$, we obtain the equality between the returns to investment:

$$(1-\delta)B\Delta^\delta(b^\star u^\star)^{-\alpha\delta}=\alpha(b^\star u^\star)^{1-\alpha}-\Phi(\tau)$$

that defines $b^\star u^\star$ as an increasing function of τ denoted $\mathcal{R}(B,\tau)\,|_{\lambda=0}$ (with $\partial\mathcal{R}(B,\tau)\,|_{\lambda=0}/\partial\tau>0$). Using $\dot{x}=\dot{b}=0$, we obtain the expression of $u^\star$:

$$u^\star=\delta+\frac{\varrho}{B\Delta^\delta\left(\mathcal{R}(B,\tau)\,|_{\lambda=0}\right)^{-\alpha\delta}}$$

which is increasing in τ and the growth rate along the BGP is:

$$g^\star=(1-\delta)B\Delta^\delta\left(\mathcal{R}(B,\tau)\,|_{\lambda=0}\right)^{-\alpha\delta}-\varrho$$

Therefore, with $\delta>0$ and $\lambda=0$ we have $\partial g^\star/\partial\tau<0$.

[21] $\alpha^{-1}(B-\lambda)[B-(2-\eta)\lambda-\varrho]>\eta\lambda(\varrho+\lambda)$ and $[B-(2-\eta)\lambda-\varrho]\times\mathcal{A}_0\Phi(\tau)>0\Rightarrow\Gamma(\bar{u}_0;\tau)>0$.

Appendix B

In this appendix, we investigate the stability properties of the BGP equilibrium for $\delta \geq 0$.

The dynamical system (A.6) may be linearized around the steady-state and becomes:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{b}(t) \\ \dot{u}(t) \end{pmatrix} = \mathcal{J} \times \begin{pmatrix} x(t) - x^\star \\ b(t) - b^\star \\ u(t) - u^\star \end{pmatrix}$$

where $\mathcal{J}$ is the Jacobian matrix evaluated at the neighbourhood of the steady-state:

$$\mathcal{J} \equiv \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}$$

with

$$\begin{aligned}
J_{11} = &\quad \eta\lambda(\varrho+\lambda)x^{\star-1} + x^\star > 0 \\
J_{12} = &\quad \tfrac{-(1-\alpha)^2(u^\star-\delta)}{(1-\delta)u^\star}\mathcal{R}(B,\tau)^{1-\alpha}x^\star/b^\star < 0 \\
J_{13} = &\quad -(1-\alpha)\mathcal{R}(B,\tau)^{1-\alpha}\left[\tfrac{1-\alpha}{1-\delta} + \alpha\tfrac{\Delta}{u^\star}\right]x^\star/u^\star < 0 \\
J_{21} = &\quad b^\star > 0 \\
J_{22} = &\quad -\alpha\delta(1-u^\star)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} - (1-\alpha)\left[1-\Delta\left(\tfrac{1}{u^\star}-1\right)\right]\mathcal{R}(B,\tau)^{1-\alpha} < 0 \\
J_{23} = &\quad -\left[1+\alpha\delta(1/u^\star-1)\right]B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}b^\star - (1-\alpha\delta)\left(\tfrac{1-\alpha}{1-\delta}\right)\mathcal{R}(B,\tau)^{1-\alpha}b^\star/u^\star < 0 \\
J_{31} = &\quad -u^\star < 0 \\
J_{32} = &\quad -u^\star/b^\star\left\{\alpha\delta(\alpha^{-1}-1+u^\star)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} + (1-\alpha)\Delta\left[\tfrac{1}{u^\star}+\tfrac{\alpha}{1-\alpha}\right]\mathcal{R}(B,\tau)^{\star 1-\alpha}\right\} < 0 \\
J_{33} = &\quad B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}\left[u^\star-\delta+\alpha\delta(1-u^\star)\right] + \alpha\Delta\left(\tfrac{1}{u^\star}-1\right)\mathcal{R}(B,\tau)^{1-\alpha} > 0
\end{aligned}$$

The determinant of the Jacobian matrix is

$$\det(\mathcal{J}) = J_{22}\left(J_{11}J_{33} - J_{13}J_{31}\right) + J_{32}\left(J_{21}J_{13} - J_{23}J_{11}\right) + J_{12}\left(J_{31}J_{23} - J_{33}J_{21}\right) < 0$$

because, under conditions (C1)-(C2) (inequality (A.11) holds),

$$\begin{aligned}
J_{11}J_{33} - J_{13}J_{31} = J_{11} \times &\left[\frac{(1-u^\star)(1-\alpha)\alpha\delta}{u^\star(1-\delta)}\mathcal{R}(B,\tau)^{1-\alpha} + (u^\star-\delta+\alpha\delta(1-u^\star))B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}\right] \\
&- \frac{(1-\alpha)(u^\star-\delta)x^\star}{u^\star(1-\delta)}\mathcal{R}(B,\tau)^{1-\alpha} > 0
\end{aligned}$$

$$\begin{aligned}
J_{21}J_{13} - J_{23}J_{11} = &\frac{b^\star}{(1-\delta)u^{\star 2}} \times \Big\{\eta\lambda(\varrho+\lambda)u^\star\Big(\delta(1-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta} + (1-\alpha)(1-\alpha\delta)\mathcal{R}(B,\tau)^{1-\alpha} \\
&+(1-\delta)(1-\alpha\delta)u^\star B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}\Big) + x^\star\Big[\alpha(1-\delta)\Big(\alpha(1-\alpha)(u^\star-(1-\alpha)\delta)\mathcal{R}(B,\tau)^{1-\alpha} \\
&+\delta u^\star x^\star B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}\Big) + \alpha\delta(1-\alpha)^2(1-\delta)\mathcal{R}(B,\tau)^{1-\alpha} \\
&+(1-\alpha\delta)(1-\delta)u^{\star 2}B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}\Big]\Big\} > 0
\end{aligned}$$

$$J_{31}J_{23} - J_{33}J_{21} = \frac{b^\star}{(1-\delta)u^\star}\left[(1-\alpha)(u^\star - \alpha\delta)\mathcal{R}(B,\tau)^{1-\alpha} + (1-\delta)\delta B\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta}\right] > 0$$

And the trace of the Jacobian matrix is

$$\text{Trace}(\mathcal{J}) = J_{11}+J_{22}+J_{33} = (u^\star-\delta)\Delta^\delta \mathcal{R}(B,\tau)^{-\alpha\delta}+\eta\lambda(\varrho+\lambda)x^{\star-1}+x^\star-\frac{(1-\alpha)(u^\star-\delta)}{(1-\delta)u^\star}\mathcal{R}(B,\tau)^{1-\alpha}$$

From equation (A.13) we have $x^\star-\frac{(1-\alpha)(u^\star-\delta)}{(1-\delta)u^\star}\mathcal{R}(B,\tau)^{1-\alpha} = (u^\star-\delta)B\Delta^\delta\mathcal{R}(B,\tau)^{-\alpha\delta}-\eta\lambda > 0$, therefore the Trace of the Jacobian matrix is positive.

Because there are two control variables (u and x) and one state-variable (b), the negative determinant and the positive trace of the Jacobian matrix imply that there are two positive eigenvalues and one negative eigenvalue. Therefore, the equilibrium is saddle-path stable.

Note that when $\delta = 0$, we obtain

$$\det(\mathcal{J}) = -(1-\alpha)Bu^\star\left(\mathcal{R}(B,\tau)\right)^{1-\alpha}\left[x^\star + \eta\lambda(\varrho+\lambda)x^{\star-1}\right] < 0$$

and, using equation (A.13) and the inequality (A.11),

$$\text{Trace}(\mathcal{J}) = 2Bu^\star - \eta\lambda + \eta\lambda(\varrho+\lambda)x^{\star-1} > 0$$