Reconsidering the impact of the environment on long-run growth when pollution influences health and agents have a finite-lifetime

Mathematical Appendix

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In this appendix, we present the calculus for obtaining the main results used in the paper. Appendix A completes the endnote 7 of the paper. References to equations without the prefix A are references to the equations in the body of the paper.

A The aggregate accumulation of human capital with growing population

With a growing population at constant exogenous rate g_p, the size of a cohort born at time t is $(\lambda_t + g_p)e^{g_p t}$: enough new people are born to replace those who die $\lambda_t e^{g_p t}$ and to obtain a net growth $g_p e^{g_p t}$. Consequently the aggregate human capital becomes:

$$H_t = \int_{-\infty}^{t} h_{s,t}(\lambda_s + g_p)e^{g_p s - \lambda_s(t-s)}ds,$$

Differentiating with respect to time we obtain:

$$\dot{H}_t = \int_{-\infty}^{t} B[1-u_{s,t}]H_{s,t}ds - \int_{-\infty}^{t} \lambda_s H_{s,t}ds + (\lambda_t + g_p)e^{g_p t}h_{t,t}$$

The last term of the equation means that a new cohort of size $(\lambda_t + g_p)e^{g_p t}$ appears with an amount of individual human capital $h_{t,t}$. The larger the size

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of this cohort, the higher the population growth. We assume that the overall human capital of a newborn $(\lambda_t + g_p)e^{g_p t}h_{t,t}$ is inherited from the dying generation and is proportional to the current aggregate stock of human capital such that $(\lambda_t + g_p)e^{g_p t}h_{t,t} = \eta H_t$. Contrary to the case where population is constant, it means that the lower the human capital inherited by each newborn, the higher is population growth: $h_{t,t,} = \frac{\eta H_t}{(\lambda_t+g_p)e^{g_p t}}$. Furthermore, the effect of population growth on aggregate human capital accumulation vanishes. If we assume that $h_{t,t}$ remains constant whatever the growth of population, a positive output growth in the long-run would be impossible: the growth rate of aggregate human capital would depend on the current time (via $e^{g_p t}$) and so it could not be constant even in the long-run.

B Derivation of the BGP equilibrium in the centralized economy

B.1 Objective and first-order conditions at the optimum

As noted by Calvo and Obstfeld (1988), the social welfare function, at time $t = 0$ is the sum of two components. The first component captures the expected utilities of agents from each of the generations to be born, measured from the moment of birth. The second component captures expected utilities of agents from each of those generations currently alive, over the remainder of their lifetimes, measured from the time $t = 0$. The planner discount rate is equal to the pure time-preference ρ to avoid problems of time consistency (see Calvo and Obstfeld (1988) for more details). Consequently, welfare at $t = 0$ is

$$W_0 = \int_0^{\infty} \left\{ \int_s^{\infty} U[c_{s,t}, \mathcal{P}_t] L_{s,t} e^{-\rho(t-s)} dt \right\} e^{-\rho s} ds + \int_{-\infty}^{0} \left\{ \int_0^{\infty} U[c_{s,t}, \mathcal{P}_t] L_{s,t} e^{-\rho t} dt \right\} ds \quad \text{(A.1)}$$

Note that the second term in the right-hand side is discounted by the planner at time 0, so we write it as $\int_{-\infty}^{0} \left\{ \int_0^{\infty} U[c_{s,t}, \mathcal{P}_t] L_{s,t} e^{-\rho(t-0)} dt \right\} e^{-\rho 0} ds$.

After changing the order of derivation, we can write (A.1) as

$$W_0 = \int_0^{\infty} \left\{ \int_{-\infty}^{t} U[c_{s,t}, \mathcal{P}_t] L_{s,t} ds \right\} e^{-\rho t} dt \quad \text{(A.2)}$$

The program of the social planner is:

$$\begin{aligned}
\max_{\substack{c_{s,t},u_{s,t},A_t,\theta_t\\ K_t,H_t,H_{s,t}}} \quad & \int_0^{\infty}\left\{\int_{-\infty}^{t} U[c_{s,t},\mathcal{P}_t]L_{s,t}ds\right\}e^{-\rho t}dt\\
\text{s.t.}\quad & \dot{K}_t=(1-\theta_t)K_t^{\alpha}[\textstyle\int_{-\infty}^{t}u_{s,t}H_{s,t}ds]^{1-\alpha}-\int_{-\infty}^{t}c_{s,t}L_{s,t}ds-\xi A_t\\
& \dot{H}_t=\textstyle\int_{-\infty}^{t}B[1-u_{s,t}]H_{s,t}ds-\int_{-\infty}^{t}\lambda_s H_{s,t}ds+\eta\lambda_t H_t\\
& H_t=\textstyle\int_{-\infty}^{t}H_{s,t}ds\\
& \mathcal{P}_t=(K_t/A_t)^{\gamma}\\
& \lambda_t=\frac{\delta\mathcal{P}_t^{\psi}}{\beta\theta_t}\\
& K_t>0, H_t>0, K_0 \text{ and } H_0 \text{ given,}
\end{aligned} \tag{10}$$

with $U(c_{s,t},\mathcal{P}_t)$ defined by equation (3) in the paper.

To solve (10), we define the Lagrangian, with $\sigma\neq 1$, as:

$$\begin{aligned}
\mathcal{L}=\frac{e^{-\rho t}}{1-1/\sigma}&\left\{\int_{-\infty}^{t}\left\{\left[c_{s,t}\left(\frac{K_t}{A_t}\right)^{-\gamma\phi}\right]^{1-1/\sigma}-1\right\}L_{s,t}ds\right\}\\
&+\pi_{1,t}\left\{(1-\theta_t)K_t^{\alpha}\left[\int_{-\infty}^{t}u_{s,t}H_{s,t}ds\right]^{1-\alpha}-\int_{-\infty}^{t}c_{s,t}L_{s,t}ds-\xi A_t\right\}\\
&+\pi_{2,t}\left\{\int_{-\infty}^{t}B[1-u_{s,t}]H_{s,t}ds-\int_{-\infty}^{t}\lambda_s H_{s,t}ds+\eta\lambda_t H_t\right\}\\
&+v_t\left\{H_t-\int_{-\infty}^{t}H_{s,t}ds\right\}
\end{aligned} \tag{A.3}$$

where π_1 and π_2 are the costate variables for an interior solution and v is the Lagrangian multiplier.

The necessary conditions are:

$$\frac{\partial\mathcal{L}}{\partial c_{s,t}}=0 \qquad \Rightarrow \qquad e^{-\rho t}c_{s,t}^{-1/\sigma}\left(\frac{K_t}{A_t}\right)^{-\gamma\phi(1-1/\sigma)}=\pi_{1,t} \tag{A.4}$$

$$\frac{\partial\mathcal{L}}{\partial u_{s,t}}=0 \qquad \Rightarrow \qquad \pi_{1,t}(1-\theta_t)(1-\alpha)K_t^{\alpha}\left(\int_{-\infty}^{t}u_{s,t}H_{s,t}ds\right)^{-\alpha}=\pi_{2,t}B \tag{A.5}$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial A_t}=0 \qquad \Rightarrow \qquad & \frac{\phi\gamma e^{-\rho t}}{A_t}\left(\frac{K_t}{A_t}\right)^{-\gamma\phi(1-1/\sigma)}\int_{-\infty}^{t}c_{s,t}^{1-1/\sigma}L_{s,t}ds-\xi\pi_{1,t}\\
&-\pi_{2,t}\frac{\partial\lambda_t}{\partial A_t}\left\{\int_{-\infty}^{t}H_{s,t}ds-\eta H_t\right\}=0
\end{aligned} \tag{A.6}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_t} = 0 \qquad \Rightarrow \qquad -\pi_{1,t} K_t^{\alpha} \left(\int_{-\infty}^{t} u_{s,t} H_{s,t} ds \right)^{1-\alpha} - \pi_{2,t} \frac{\partial \lambda_t}{\partial \theta_t} \left\{ \int_{-\infty}^{t} H_{s,t} ds - \eta H_t \right\} = 0 \quad \text{(A.7)}$$

$$\frac{\partial \mathcal{L}}{\partial K_t} = -\dot{\pi}_{1,t} \qquad \Rightarrow \qquad \frac{-\phi\gamma e^{-\rho t}}{K_t} \left(\frac{K_t}{A_t} \right)^{-\gamma\phi(1-1/\sigma)} \int_{-\infty}^{t} c_{s,t}^{1-1/\sigma} L_{s,t} ds + \pi_{1,t} \alpha (1-\theta_t) K_t^{\alpha-1} \left(\int_{-\infty}^{t} u_{s,t} H_{s,t} ds \right)^{1-\alpha} - \pi_{2,t} \frac{\partial \lambda_t}{\partial K_t} \left\{ \int_{-\infty}^{t} H_{s,t} ds - \eta H_t \right\} = -\dot{\pi}_{1,t} \quad \text{(A.8)}$$

$$\frac{\partial \mathcal{L}}{\partial H_t} = -\dot{\pi}_{2,t} \qquad \Rightarrow \qquad \pi_{2,t} \eta \lambda_t + v_t = -\dot{\pi}_{2,t} \quad \text{(A.9)}$$

$$\frac{\partial \mathcal{L}}{\partial H_{s,t}} = 0 \qquad \Rightarrow \qquad \pi_{1,t}(1-\alpha)(1-\theta_t) K_t^{\alpha} \left(\int_{-\infty}^{t} u_{s,t} H_{s,t} ds \right)^{-\alpha} u_{s,t} + \pi_{2,t} \left\{ B[1-u_{s,t}] - \lambda_s \right\} - v_t = 0 \quad \text{(A.10)}$$

$$\lim_{t\to\infty} \pi_{1,t} K_t e^{-\rho t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \pi_{2,t} H_t e^{-\rho t} = 0 \quad \text{(A.11)}$$

with $\dfrac{\partial \lambda_t}{\partial A_t} = -\psi\gamma\lambda_t / A_t$, $\dfrac{\partial \lambda_t}{\partial K_t} = \psi\gamma\lambda_t / K_t$ and $\dfrac{\partial \lambda_t}{\partial \theta_t} = -\psi\gamma\lambda_t / \theta_t$.

First, (A.4) implies that $c_{s,t}$ is independent from s: $c_{s,t} = c_t$. Consequently, because $\int_{-\infty}^{t} L_{s,t} ds = 1$ and $C_t = \int_{-\infty}^{t} c_{s,t} L_{s,t} ds$, we have

$$c_{s,t} = C_t$$

Based on (A.2), we write the social welfare function as

$$W_0 = \int_0^{\infty} U[C_t, \mathcal{P}_t] e^{-\rho t} dt. \quad \text{(A.12)}$$

From equations (A.5) and (A.10), we obtain

$$B - \lambda_s = \frac{v_t}{\pi_{2,t}} \quad \text{(A.13)}$$

Consequently, λ_s is independent from s:

$$\lambda_s = \lambda_t$$

(A.10) means that $u_{s,t}$ is also independent from s:

$$u_{s,t} = u_t$$

Using (A.9) and the previous results, we have:

$$\frac{\dot{\pi_{2,t}}}{\pi_{2,t}} = \lambda_t(1-\eta) - B \tag{A.14}$$

Extracting $-\phi\gamma e^{-\rho t}$ in (A.6) and introducing it in (A.8), using the previous results and simplifying, we obtain:

$$\frac{\dot{\pi_{1,t}}}{\pi_{1,t}} = \xi\left(\frac{A_t}{K_t}\right) - \alpha(1-\theta_t)K_t^{\alpha-1}(u_t H_t)^{1-\alpha} \tag{A.15}$$

Differentiating (A.5) with respect to time and using the previous results, it becomes:

$$\frac{\dot{u_t}}{u_t} = \alpha^{-1}\left(\frac{\dot{\pi_{1,t}}}{\pi_{1,t}} - \frac{\dot{\pi_{2,t}}}{\pi_{2,t}} - \frac{\dot{\theta}_t}{1-\theta_t}\right) - \frac{\dot{H}_t}{H_t} + \frac{\dot{K}_t}{K_t} \tag{A.16}$$

Differentiating (A.4) with respect to time and recalling that $\mathcal{P}_t = (K_t/A_t)^\gamma$, we obtain:

$$\frac{\dot{C}_t}{C_t} = -\sigma\frac{\dot{\pi_{1,t}}}{\pi_{1,t}} - \sigma\rho - \phi\left(\sigma-1\right)\frac{\dot{\mathcal{P}}_t}{\mathcal{P}_t} \tag{A.17}$$

Furthermore, we also have

$$\frac{\dot{K}_t}{K_t} = (1-\theta_t)K_t^{\alpha-1}(u_t H_t)^{1-\alpha} - \frac{C_t}{K_t} - \xi\mathcal{P}_t^{-1/\gamma} \tag{A.18}$$

and

$$\frac{\dot{H}_t}{H_t} = B(1-u_t) - (1-\eta)\lambda_t \tag{A.19}$$

B.2 The balanced growth path

From equations (6) and (8) in the text, the steady state in this economy is a balanced growth path where H, K, A, C and Y grow at a common rate and where the intersectoral allocation of human capital u and the part of the aggregate final output used to provide public-health services θ are constant.

We define $x \equiv C/K$ and $b \equiv H/K$. Along the BGP, x, b, $\mathcal{P}$ and λ are also constant.

From (A.17), (A.18) and (A.15), $\dot{x} = 0$ implies

$$\sigma\alpha(1-\theta^\star)(b^\star u^\star)^{1-\alpha} - \sigma\rho = (1-\theta^\star)(b^\star u^\star)^{1-\alpha} - x^\star - (1-\sigma)\xi\mathcal{P}^{\star -1/\gamma} \quad \text{(A.20)}$$

and from (A.18), (A.19) and (A.15), $\dot{b} = 0$ implies

$$B(1-u^\star) - \lambda^\star(1-\eta) = (1-\theta^\star)(b^\star u^\star)^{1-\alpha} - x^\star - \xi\mathcal{P}^{\star -1/\gamma} \quad \text{(A.21)}$$

where a star denotes a variable along the BGP.

Furthermore, with $\dot{u} = \dot{\theta} = 0$, (A.15), (A.14) and (A.16) give

$$\alpha(1-\theta^\star)(b^\star u^\star)^{1-\alpha} - \xi\mathcal{P}^{\star -1/\gamma} = B - \lambda^\star(1-\eta) \quad \text{(A.22)}$$

that is the return to the accumulation of physical capital equal the return to accumulation of human capital. Subtracting (A.20) and (A.21) and using (A.22), we obtain the value of the allocation of human capital to production in the long-run

$$u^\star = \frac{\sigma\rho}{B} + (1-\sigma)\frac{B - \lambda^\star(1-\eta)}{B}. \quad (11)$$

Since $u^\star \in]0,1[$, we always have $\sigma\rho + (1-\sigma)B > 0$.

Using (A.20) and (A.22), we have

$$x^\star = \frac{1-\alpha\sigma}{\alpha}\left[B + \xi\mathcal{P}^{\star -1/\gamma} - \lambda^\star(1-\eta)\right] + \sigma\rho - (1-\sigma)\xi\mathcal{P}^{\star -1/\gamma} \quad \text{(A.23)}$$

Using (A.5), (A.7), (11) and recalling that $\lambda^\star = \frac{\delta\mathcal{P}^{\star\psi}}{\beta\theta^\star}$, we obtain a relation between $\theta^\star$ the part of health care expenditures and the net pollution flow along the BGP $\mathcal{P}^\star$:

$$\theta^{\star 2} + \frac{(\sigma-\alpha)\varphi\mathcal{P}^{\star\psi}}{1-\alpha}\theta^\star = \varphi\mathcal{P}^{\star\psi}$$

with $\varphi \equiv \frac{(1-\alpha)(1-\eta)\delta}{\beta[\sigma\rho+(1-\sigma)B]} > 0$ since $u^\star$ defined by (11) must be positive. Consequently, the higher the net flow of pollution in the long-run, the higher the part of the public health-care expenditures in GDP. Solving this relation gives

$$\theta^\star = \frac{-(\sigma-\alpha)\varphi\mathcal{P}^{\star\psi} + \sqrt{(\sigma-\alpha)^2\varphi^2\mathcal{P}^{\star 2\psi} + 4(1-\alpha)^2\varphi\mathcal{P}^{\star\psi}}}{2(1-\alpha)} \quad \text{(A.24)}$$

Consequently, the probability of death along the BGP depends positively on the net flow of pollution:

$$\lambda^\star = \frac{\delta \mathcal{P}^{\star\psi}}{\beta\theta^\star} = \frac{2(1-\alpha)\delta}{\beta\left[-(\sigma-\alpha)\varphi + \sqrt{(\sigma-\alpha)^2\varphi^2 + 4(1-\alpha)^2\varphi/\mathcal{P}^{\star\psi}}\right]} \equiv \underset{+}{\Lambda(\mathcal{P}^\star)}$$

Note that $\theta^\star$ is always lower than unity and that $\lim_{\mathcal{P}^\star\to 0}\Lambda(\mathcal{P}^\star) = 0$ and $\lim_{\mathcal{P}^\star\to+\infty}\Lambda(\mathcal{P}^\star) = +\infty$.

Finally, using the value of $u^\star$, we see that the expression of the growth rate along the BGP depends negatively on the long-run flow of pollution:

$$g^\star = \sigma B - \sigma\rho - \sigma\Lambda(\mathcal{P}^\star)(1-\eta)$$

Using (A.5), (A.4) and (A.6), we can define $\mathcal{P}^\star$ as

$$\gamma\left(\frac{1-\alpha}{\alpha}\right)\left[\phi + \frac{(1-\eta)\Lambda(\mathcal{P}^\star)}{\rho}\right]\left[B + \mathcal{P}^{\star-1/\gamma} - (1-\eta)\Lambda(\mathcal{P}^\star)\right] + \gamma\phi\rho - \xi\mathcal{P}^{\star-1/\gamma} = 0 \tag{12}$$

Note that $\mathcal{P}^\star$ is constant along the BGP and therefore we have verified that the environmental quality is constant in the long-run.

C Corner solutions for u

In this section we investigate the possibility of corner solutions for u in the centralized equilibrium. Because $\lim_{u\to 0}\partial Y/\partial u = \infty$, $u = 0$ is not sutainable as an equilibrium.

For $u = 1$ we have to solve the program of the social planner expliciting the condition $u \le 1$. Consequently we write

$$\max_{\substack{c_{s,t},u_{s,t},A_t,\theta_t\\ K_t,H_t,H_{s,t}}} \int_0^\infty \left\{\int_{-\infty}^t U[c_{s,t},\mathcal{P}_t]L_{s,t}ds\right\} e^{-\rho t}dt$$

$$\begin{aligned}
\text{s.t.}\quad & \dot{K}_t = (1-\theta_t)K_t^\alpha\left[\int_{-\infty}^t u_{s,t}H_{s,t}ds\right]^{1-\alpha} - \int_{-\infty}^t c_{s,t}L_{s,t}ds - \xi A_t\\
& \dot{H}_t = \int_{-\infty}^t B[1-u_{s,t}]H_{s,t}ds - \int_{-\infty}^t \lambda_s H_{s,t}ds + \eta\lambda_t H_t\\
& H_t = \int_{-\infty}^t H_{s,t}ds\\
& \mathcal{P}_t = (K_t/A_t)^\gamma\\
& \lambda_t = \frac{\delta\mathcal{P}_t^\psi}{\beta\theta_t}\\
& 0 \le 1 - u_{s,t}\\
& K_t > 0, H_t > 0, K_0 \text{ and } H_0 \text{ given,}
\end{aligned}$$

with $U(c_{s,t},\mathcal{P}_t)$ defined by equation (3) in the paper.

The Lagrangian, with $\sigma \neq 1$, is:

$$\mathcal{L} = \frac{e^{-\rho t}}{1-1/\sigma}\left\{\int_{-\infty}^{t}\left\{\left[c_{s,t}\left(\frac{K_t}{A_t}\right)^{-\gamma\phi}\right]^{1-1/\sigma}-1\right\}L_{s,t}ds\right\}$$
$$+\pi_{1,t}\left\{(1-\theta_t)K_t^{\alpha}\left[\int_{-\infty}^{t}u_{s,t}H_{s,t}ds\right]^{1-\alpha}-\int_{-\infty}^{t}c_{s,t}L_{s,t}ds-\xi A_t\right\}$$
$$+\pi_{2,t}\left\{\int_{-\infty}^{t}B[1-u_{s,t}]H_{s,t}ds-\int_{-\infty}^{t}\lambda_s H_{s,t}ds+\eta\lambda_t H_t\right\}$$
$$+v_t\left\{H_t-\int_{-\infty}^{t}H_{s,t}ds\right\}+\int_{-\infty}^{t}\zeta_{s,t}(1-u_{s,t})ds \quad (A.25)$$

where $\zeta_{s,t}\geq 0$ is the Lagrangian multiplier associated to $0\leq 1-u_{s,t}$.

The necessary conditions are:

$$\frac{\partial\mathcal{L}}{\partial c_{s,t}}=0 \quad\Rightarrow\quad e^{-\rho t}c_{s,t}^{-1/\sigma}\left(\frac{K_t}{A_t}\right)^{-\gamma\phi(1-1/\sigma)}=\pi_{1,t} \quad (A.26)$$

$$\frac{\partial\mathcal{L}}{\partial u_{s,t}}=0 \quad\Rightarrow\quad \pi_{1,t}(1-\theta_t)(1-\alpha)K_t^{\alpha}\left(\int_{-\infty}^{t}u_{s,t}H_{s,t}ds\right)^{-\alpha}=\pi_{2,t}B+\frac{\zeta_{s,t}}{H_{s,t}} \quad (A.27)$$

$$\frac{\partial\mathcal{L}}{\partial A_t}=0 \quad\Rightarrow\quad \frac{\phi\gamma e^{-\rho t}}{A_t}\left(\frac{K_t}{A_t}\right)^{-\gamma\phi(1-1/\sigma)}\int_{-\infty}^{t}c_{s,t}^{1-1/\sigma}L_{s,t}ds-\xi\pi_{1,t}$$
$$-\pi_{2,t}\frac{\partial\lambda_t}{\partial A_t}\left\{\int_{-\infty}^{t}H_{s,t}ds-\eta H_t\right\}=0 \quad (A.28)$$

$$\frac{\partial\mathcal{L}}{\partial\theta_t}=0 \quad\Rightarrow\quad -\pi_{1,t}K_t^{\alpha}\left(\int_{-\infty}^{t}u_{s,t}H_{s,t}ds\right)^{1-\alpha}$$
$$-\pi_{2,t}\frac{\partial\lambda_t}{\partial\theta_t}\left\{\int_{-\infty}^{t}H_{s,t}ds-\eta H_t\right\}=0 \quad (A.29)$$

$$\frac{\partial\mathcal{L}}{\partial K_t}=-\dot{\pi}_{1,t} \quad\Rightarrow\quad \frac{-\phi\gamma e^{-\rho t}}{K_t}\left(\frac{K_t}{A_t}\right)^{-\gamma\phi(1-1/\sigma)}\int_{-\infty}^{t}c_{s,t}^{1-1/\sigma}L_{s,t}ds$$
$$+\pi_{1,t}\alpha(1-\theta_t)K_t^{\alpha-1}\left(\int_{-\infty}^{t}u_{s,t}H_{s,t}ds\right)^{1-\alpha}$$
$$-\pi_{2,t}\frac{\partial\lambda_t}{\partial K_t}\left\{\int_{-\infty}^{t}H_{s,t}ds-\eta H_t\right\}=-\dot{\pi}_{1,t} \quad (A.30)$$

$$\frac{\partial \mathcal{L}}{\partial H_t} = -\dot{\pi}_{2,t} \qquad \Rightarrow \qquad \pi_{2,t}\eta\lambda_t + v_t = -\dot{\pi}_{2,t} \tag{A.31}$$

$$\frac{\partial \mathcal{L}}{\partial H_{s,t}} = 0 \qquad \Rightarrow \qquad \pi_{1,t}(1-\alpha)(1-\theta_t)K_t^{\alpha}\left(\int_{-\infty}^{t} u_{s,t}H_{s,t}ds\right)^{-\alpha} u_{s,t} + \pi_{2,t}\left\{B[1-u_{s,t}] - \lambda_s\right\} - v_t = 0 \tag{A.32}$$

$$\lim_{t\to\infty} \pi_{1,t}K_t e^{-\rho t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \pi_{2,t}H_t e^{-\rho t} = 0 \tag{A.33}$$

with $\dfrac{\partial \lambda_t}{\partial A_t} = -\psi\gamma\lambda_t/A_t$, $\dfrac{\partial \lambda_t}{\partial K_t} = \psi\gamma\lambda_t/K_t$ and $\dfrac{\partial \lambda_t}{\partial \theta_t} = -\psi\gamma\lambda_t/\theta_t$.

Considering the case where the constraint $0 \leq 1-u_{s,t}$ is binding, equations (A.27), (A.29), (A.30) and (A.32) become:

$$\pi_{1,t}(1-\theta_t)(1-\alpha)K_t^{\alpha}H_t^{-\alpha} = \pi_{2,t}B + \frac{\zeta_{s,t}}{H_{s,t}} \tag{A.34}$$

$$-\pi_{1,t}K_t^{\alpha}H_t^{1-\alpha} - \pi_{2,t}\frac{\partial \lambda_t}{\partial \theta_t}\left\{\int_{-\infty}^{t} H_{s,t}ds - \eta H_t\right\} = 0 \tag{A.35}$$

$$\frac{-\phi\gamma e^{-\rho t}}{K_t}\left(\frac{K_t}{A_t}\right)^{-\gamma\phi(1-1/\sigma)} \int_{-\infty}^{t} c_{s,t}^{1-1/\sigma}L_{s,t}ds + \pi_{1,t}\alpha(1-\theta_t)K_t^{\alpha-1}H_t^{1-\alpha} - \pi_{2,t}\frac{\partial \lambda_t}{\partial K_t}\left\{\int_{-\infty}^{t} H_{s,t}ds - \eta H_t\right\} = -\dot{\pi}_{1,t} \tag{A.36}$$

$$\pi_{1,t}(1-\alpha)(1-\theta_t)K_t^{\alpha}H_t^{-\alpha} - \pi_{2,t}\lambda_s = v_t \tag{A.37}$$

First, (A.26) implies that $c_{s,t}$ is independent from s: $c_{s,t} = c_t = C_t$. Second, equation (A.37) implies that λ_s is independent from s:

$$\lambda_s = \lambda_t$$

From (A.34) and (A.35),

$$\frac{\zeta_{s,t}/H_{s,t}}{\pi_{2,t}} = (1-\alpha)(1-\eta)\left(\frac{1}{\theta_t} - 1\right)\psi\gamma\lambda_t - B \tag{A.38}$$

From (A.34) and (A.37),

$$B + \frac{\zeta_{s,t}/H_{s,t}}{\pi_{2,t}} - \frac{\delta\mathcal{P}_t^{\psi}}{\beta\theta_t} = \frac{v_t}{\pi_{2,t}} \tag{A.39}$$

Using (A.31) and the two previous equations, we obtain

$$\frac{\dot{\pi_{2,t}}}{\pi_{2,t}} = \left[1 - (1-\alpha)\psi\gamma\left(\frac{1}{\theta_t} - 1\right)\right](1-\eta)\lambda_t \tag{A.40}$$

The first-order conditions enable us to write:

$$\frac{\dot{\pi_{1,t}}}{\pi_{1,t}} = \xi\mathcal{P}_t^{-1/\gamma} - \alpha(1-\theta_t)K_t^{\alpha-1}H_t^{1-\alpha} \tag{A.41}$$

Furthermore, (A.17) and (A.18) with $u = 1$ give $\frac{\dot{C}_t}{C_t}$ and $\frac{\dot{K}_t}{K_t}$, respectively. Finally,

$$\frac{\dot{H}_t}{H_t} = -(1-\eta)\lambda_t \leq 0$$

Along the balanced growth path, H, K, A, C and Y evolve at a common rate and θ, $\mathcal{P}$ and therefore λ are constant. If $\lambda^\star > 0$ (the $\star$ denotes steady-state), the aggregate human capital accumulation is negative and H, K, A, C, Y decrease towards 0: the economy vanishes and $u = 1$ can not be sustained as an equilibrium. If $\lambda^\star = 0$, $\dot{H}/H = 0$ and there could be a steady-state equilibrium with no growth.

From (A.35), $\pi_{1,t}/\pi_{2,t}$ is constant in the steady-state, therefore (A.40) and (A.41) give

$$\left[1 - (1-\alpha)\psi\gamma\left(\frac{1}{\theta^\star} - 1\right)\right](1-\eta)\lambda^\star = \xi\mathcal{P}^{\star-1/\gamma} - \alpha(1-\theta^\star)b^{\star 1-\alpha} \tag{A.42}$$

Furthermore, using (A.42), $\dot{C}/C = \dot{H}/H$ implies

$$(1-\alpha)\psi\gamma\left(\frac{1}{\theta^\star} - 1\right)(1-\eta)\lambda^\star = \left(1 - \frac{1}{\sigma}\right)(1-\eta)\lambda^\star + \rho \tag{A.43}$$

Recalling that $\lambda^\star = \frac{\delta\mathcal{P}^{\star\psi}}{\beta\theta^\star}$, this equation is a second-order equation in $\theta^\star$ whose the positive solution is an increasing function of $\mathcal{P}^\star$:

$$\theta^\star = \frac{-\left(1 - \frac{1}{\sigma} + (1-\alpha)\psi\gamma\right)(1-\eta)\delta\mathcal{P}^{\star\psi}}{2\beta\rho} + \frac{\sqrt{\left(\left(1 - \frac{1}{\sigma} + (1-\alpha)\psi\gamma\right)(1-\eta)\delta\mathcal{P}^{\star\psi}\right)^2 + 4\beta\rho(1-\alpha)(1-\eta)\psi\gamma\delta\mathcal{P}^{\star\psi}}}{2\beta\rho} \tag{A.44}$$

Consequently, we obtain

$$\lambda^\star \equiv \underset{+}{\Lambda(\mathcal{P}^\star)} = \sqrt{\delta \mathcal{P}^{\star\psi}} \left[\frac{-\left(1-\frac{1}{\sigma}+(1-\alpha)\psi\gamma\right)(1-\eta)\sqrt{\delta\mathcal{P}^{\star\psi}}}{2\rho} + \frac{\sqrt{\left(\left(1-\frac{1}{\sigma}+(1-\alpha)\psi\gamma\right)(1-\eta)\right)^2 \delta\mathcal{P}^{\star\psi} + 4\beta\rho(1-\alpha)(1-\eta)\psi\gamma}}{2\rho}\right]^{-1} \tag{A.45}$$

with $\lim_{\mathcal{P}^\star \to +\infty} \Lambda(\mathcal{P}^\star) = +\infty$.

To have $\lambda^\star = 0$, it is required that $\mathcal{P}^\star = 0$. Because $\mathcal{P}^\star = \left(\frac{K^\star}{A^\star}\right)^\gamma$ and $A^\star$ is bounded, we conclude that $K^\star = 0$. However, at the same time $K^\star$ must be positive. Consequently, $\lambda^\star = 0$ is not possible and $u = 1$ can not be sustained as an equilibrium.

D Existence and unicity of $u_d^\star$

$u_d^\star$ is defined by

$$\Gamma(u_d^\star) \equiv \left[B\left(\frac{\sigma}{1-z^H} - 1 + u_d^\star\right) + (1-\sigma-\eta)\mathcal{L}(\tau) - \sigma\rho\right] \times \left\{\left[u_d^\star + \frac{\mathcal{A}+z^H}{1-z^H}\right] B - \mathcal{L}(\tau)\left[\mathcal{A}+\eta\right] + \mathcal{A}\xi\left[\chi\tau\right]^{\frac{1}{1+\gamma}}\right\} - \mathcal{L}(\tau)\left[\sigma\rho + \sigma\mathcal{L}(\tau) + (1-\sigma)\frac{B}{1-z^H}\right] = 0$$

with $\mathcal{A} \equiv \alpha^{-1}(1-\theta) - 1$. Recall that θ is defined by equation (A.24) with $\mathcal{P}^\star$ replaced by $\mathcal{P}$. Since $\mathcal{P}$ negatively depends on τ (see equation 16 in the paper), θ is a decreasing function of τ and $\mathcal{A}$ increases with τ.

It is straightforward that $\Gamma(u_d^\star)$ is an increasing function of $u_d^\star$. If $\mathcal{A} \geq 0$ (that is $\theta \leq 1-\alpha$) and $\sigma \geq 1-\eta$, $\Gamma(u_d^\star)$ is an increasing function of τ because $\mathcal{L}(\tau)$ decreases with τ. Consequently, from the implicit function theorem, $u_d^\star$ is a decreasing function of τ. $\Gamma(u_d^\star) = 0$ defines a unique value for $u_d^\star \in]0,1[$ if $\Gamma(0) < 0$ and $\Gamma(1) > 0$. These two conditions define $\underline{\tau}$ and $\bar{\tau}$ such that $\Gamma(0)|_{\tau=\bar{\tau}} = 0$ and $\Gamma(1)|_{\tau=\underline{\tau}} = 0$.

The growth rate in the market economy is

$$g_d^\star = B\left[1 - u_d^\star(\tau)\right] - (1-\eta)\frac{\delta}{\beta\theta}\left[\chi\tau\right]^{\frac{-\gamma\psi}{1+\gamma}}.$$

Since $g_d^\star$ increases with τ when $\theta \leq 1-\alpha$, $\sigma \geq 1-\eta$ and $\tau \in]\underline{\tau}, \bar{\tau}[$, $g_d^\star > 0$ requires that $\tau > \hat{\tau} > \underline{\tau}$ with $\hat{\tau}$ such that $B\left[1-u_d^\star(\hat{\tau})\right] = (1-\eta)\frac{\delta}{\beta\theta}\left[\chi\hat{\tau}\right]^{\frac{-\gamma\psi}{1+\gamma}}$.

E The case where θ is chosen at its optimal value

If we assume that the government fixes θ in the market economy at its optimal level, we have:

$$\theta = \frac{-(\sigma-\alpha)\varphi\mathcal{P}^{\psi} + \sqrt{(\sigma-\alpha)^2\varphi^2\mathcal{P}^{2\psi} + 4(1-\alpha)^2\varphi\mathcal{P}^{\psi}}}{2(1-\alpha)}$$

with $\mathcal{P} = [\chi\tau]^{\frac{-\gamma\psi}{1+\gamma}}$. It is straightforward that θ is a decreasing function of τ. Furthermore, the probability of death becomes

$$\lambda = \frac{\delta\mathcal{P}^{\psi}}{\beta\theta} = \frac{2(1-\alpha)\delta}{\beta\left[-(\sigma-\alpha)\varphi + \sqrt{(\sigma-\alpha)^2\varphi^2 + 4(1-\alpha)^2\varphi/\mathcal{P}^{\psi}}\right]}$$

which increases with $\mathcal{P}$ and so is a decreasing function of τ.

Finally, when θ is chosen at its optimal value, $\mathcal{A} \equiv \alpha^{-1}(1-\theta) - 1$ increases with τ. Since $b_d^{\star}u_d^{\star} > 0$, from equation (26) in the paper, $B - \mathcal{L}(\tau) + \xi\,[\chi\tau]^{\frac{1}{1+\gamma}} > 0$. Consequently, if $\theta \leq 1-\alpha$ and $\sigma \geq 1-\eta$, $\Gamma(u_d^{\star})$ defined in section C remains an increasing function of τ.

References

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